Prophet Inequalities and Online Combinatorial Optimization

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Overview

1. Prophets and Secretaries
   - The Secretary Problem
   - The Prophet Inequality Problem
   - Selecting Multiple Values
2. Online Combinatorial Optimization
   - Primer on Mathematical Programming
   - Online Contention Resolution Schemes
3. Equivalence via LP Duality
4. Variations and Open Problems

Secretary Problem (1/3)

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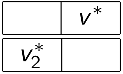

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- Optimal strategy: $r \approx \frac{n}{e}$. Then, $\Pr[\text{We select } v^*] \geq \frac{1}{e}$, and this bound is tight [Lin61; Dyn63].

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- *Adversarial order* + *Arbitrary values* $\implies \Pr[\text{We select } v^*]$ arbitrarily small.
- Assume adversarial order, but $v_i \sim D_i$, where D_i is known $\implies$ *Prophet Inequality Problem*.

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- Compare against $\mathbb{E}\left[\max_{i=1}^{n} X_i\right]$.
- $\exists$ algorithm s.t. $\mathbb{E}[\text{ALG}] \geq \frac{1}{2}\,\mathbb{E}\left[\max_{i=1}^{n} X_i\right]$, and no algorithm can achieve better competitive ratio [KS77].

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Set threshold T based on $D_1, \ldots, D_n$. Accept the first $X_i \geq T$.

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  2. Set $T = \frac{1}{2}\,\mathbb{E}[X^*]$ [KW12].

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&\geq \frac{1}{2} \mathbb{E}[X^*]
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  for some small $\varepsilon > 0$.
- For every algorithm, $\mathbb{E}[\text{ALG}] = 1$, regardless of which element it picks.
- Expected value of the prophet is

$$\mathbb{E}[X^*] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1-\varepsilon) = 2 - \varepsilon.$$

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$\mathcal{A}$: Adaptive-Threshold Algorithm

$\forall i \in [n]$, at step i, set threshold T_i, based on $D_1, \ldots, D_n$ and $X_1, \ldots, X_{i-1}$. Accept every $X_i \geq T_i$ until k values selected.

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- Simple algebra shows that

$$\mathbb{E}\left[\sum_{i \in S_T} X_i\right] \geq \left(1 - \frac{2\delta}{k}\right) OPT = \left(1 - \sqrt{\frac{8 \log k}{k}}\right) OPT.$$

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Adaptive-threshold algorithms can do better:

- $1 - \frac{1}{\sqrt{k+3}}$ [Ala14], is asymptotically tight.
- Tight competitive ratio for every $k \geq 1$ [JMZ22] (complicated LP duality argument).

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- Examples:
  1. Matroids: Uniform ($\mathcal{F} = \{S \subseteq [n] \mid |S| \leq k\}$), Graphic ($[n] \to$ edges, $\mathcal{F} \to$ forests), Vector ($[n] \to$ vectors, $\mathcal{F} \to$ lin. ind. vectors), etc.
  2. Matchings: Given $G = (V, E)$, $[n] \to E$ and $\mathcal{F} \to$ matchings in G.
  3. Knapsack: Given sizes $s_i \in [0, 1]$ for each X_i, $\mathcal{F} = \left\{S \subseteq [n] \,\middle|\, \sum_{i \in S} s_i \leq 1\right\}$.

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Matroid Prophet Inequality Theorem [KW12]

For every matroid $\mathcal{M}$, $\exists$ an algorithm for the *matroid prophet inequality* problem that returns an independent set S s.t.

$$\mathbb{E}\left[\sum_{i\in S}X_i\right]\ge\frac{1}{2}\cdot\mathbb{E}\left[\max_{S\in\mathcal{F}}\sum_{i\in S}X_i\right].$$

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A c-selectable *Contention Resolution Scheme (CRS)* is an algorithm which receives a point $\boldsymbol{x} \in \mathcal{P}$ as input and returns an independent set $S \in \mathcal{F}$ which contains every $i \in N$ with probability at least $c \cdot x_i$.

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$$\begin{aligned} \Pr\left[i \in S \mid i \in R\right] &= \Pr\left[i \in R' \mid i \in R\right] \cdot \Pr\left[1, \ldots, i-1 \notin R'\right] \\ &\geq \Pr\left[i \in R' \mid i \in R\right] \cdot \Pr\left[R' = \emptyset\right] \\ &= \frac{1}{4}. \end{aligned}$$

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- Tight: Consider $x_1 = 1 - \varepsilon$ and $x_2 = \varepsilon$ for small $\varepsilon > 0$. $\Pr[2 \in S \mid 2 \in R] = 1 - x_1 \Pr[1 \in S \mid 1 \in R] = 1 - \Pr[1 \in S \mid 1 \in R] + o(1)$.

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- Algorithm includes $\{i\} \in \mathcal{F}'$ w.p. $\frac{1-e^{-x_i}}{x_i}$. Extends to partition and transversal matroids.

Overview

From OCRS to Prophet Inequality

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- Assume a c-selectable OCRS π for some $\mathcal{F}$. Let $x_i = \Pr\left[i \in \arg\max_{I \in \mathcal{F}} \sum_{j \in I} X_j\right]$, and

$$v_i(x_i) = \mathbb{E}\left[X_i \mid X_i\text{'s value is in its top } x_i \text{ quantile}\right] \quad \text{(ex-ante PI)}.$$

Then, $\boldsymbol{x} \in \mathcal{P}$, and $OPT = \mathbb{E}\left[\max_{I \in \mathcal{F}} \sum_{j \in I} X_j\right] \le \sum_{i \in N} x_i v_i(x_i)$.

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- Essentially a reduction to Bernoulli r.v.'s.

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- If $\textsc{Primal}$ has value $\geq c \implies \exists c-$selectable OCRS.
- By strong LP duality, suffices to show $\textsc{Dual}$ has value $\geq c$. Show that $\forall \boldsymbol{y} \geq 0$ s.t. $\sum_{i \in N} x_i y_i = 1$,

$$\exists \phi \in \Phi \text{ s.t. } \sum_{i \in N} q_{i,\phi} y_i \geq c.$$

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- Assuming c-competitive (ex-ante) PI $\implies \exists \phi \in \Phi$ s.t. $\mathbb{E}[\phi] \geq c$.
- But, $\mathbb{E}[\phi] = \sum_{i \in N} q_{i,\phi} y_i$, by linearity of expectation.
- Thus, $\sum_{i \in N} q_{i,\phi} y_i \geq c$.

Overview

Prophet Secretary and I.I.D. Setting

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- *I.I.D. Prophet Inequality* problem: $\exists \approx 0.7451$-competitive ratio algorithm and it's tight [Cor+17].

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3. Optimal OCRSs for matching constraints.

and more...

QUESTIONS ?

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- One can give a simple fixed-threshold algorithm for this setting, which achieves a $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-competitive ratio.
- **Idea:** Select a threshold T such that the expected number of values $\geq T$ are $k - \delta$ for some δ.
- Since the realizations of the X_i's are independent, for an appropriately chosen δ, one can show that the number of realizations that are at least T are between $k - 2\delta$ and k, with high probability (Hoeffding bound).
- For fixed realizations, let $S_T = \{i \in [n] \mid X_i \geq T\}$. Then

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- Since $|S_T| \geq k - 2\delta$, our revenue is at least $(k - 2\delta)T$.

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- Let S^* be the optimal set selected by the prophet. Then

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- For $\delta = \sqrt{2k \log k}$, we get

$$\sum_{i \in S_T} X_i \geq \left(1 - \frac{2\delta}{k}\right) OPT = \left(1 - \sqrt{\frac{8 \log k}{k}}\right) OPT.$$

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