On Submodular Prophet Inequalities and Correlation Gap

Vasilis Livanos

Joint work with Chandra Chekuri

University of Illinois at Urbana-Champaign

September 23rd, 2021

Prophet Inequality Problem

$X_1 = 1$, w.p. 1

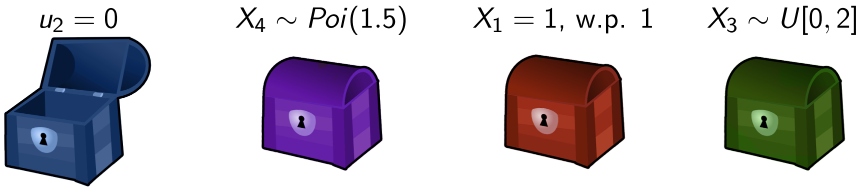
$X_2 = \begin{cases} 9, \text{w.p.} \frac{1}{9} \\ 0, \text{w.p.} \frac{8}{9} \end{cases}$

$X_3 \sim U[0, 2]$

$X_4 \sim Poi(1.5)$

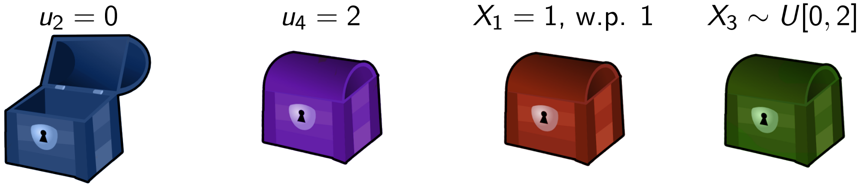
- Given independent r.v.'s $X_1, \ldots, X_n \sim D_1, \ldots, D_n$.
- $X_1, \ldots, X_n$ arrive in *adversarial order*. At step i, see realization u_i of X_i.
- Decide *immediately* and *irrevocably* to select or discard u_i.
- **Goal:** Select highest possible *single value*, compare against offline optimal $\mathbb{E}[\max_i X_i]$.

Prophet Inequality Problem



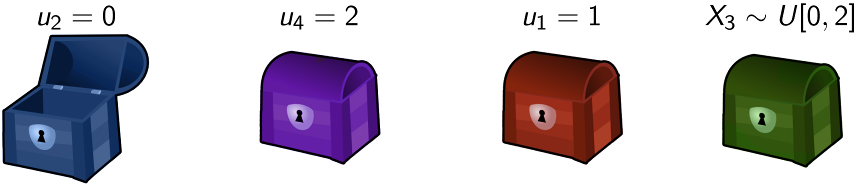
- Given independent r.v.'s $X_1, \ldots, X_n \sim D_1, \ldots, D_n$.
- $X_1, \ldots, X_n$ arrive in *adversarial order*. At step i, see realization u_i of X_i.
- Decide *immediately* and *irrevocably* to select or discard u_i.
- **Goal:** Select highest possible *single value*, compare against offline optimal $\mathbb{E}[\max_i X_i]$.

Prophet Inequality Problem



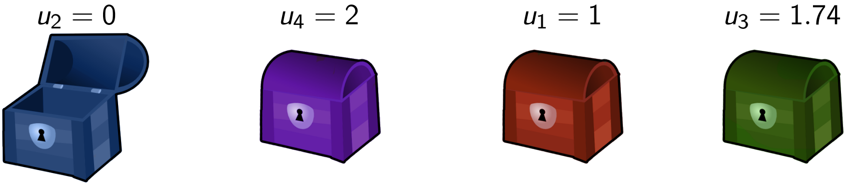
- Given independent r.v.'s $X_1, \ldots, X_n \sim D_1, \ldots, D_n$.
- $X_1, \ldots, X_n$ arrive in *adversarial order*. At step i, see realization u_i of X_i.
- Decide *immediately* and *irrevocably* to select or discard u_i.
- **Goal:** Select highest possible *single value*, compare against offline optimal $\mathbb{E}[\max_i X_i]$.

Prophet Inequality Problem



- Given independent r.v.'s $X_1, \ldots, X_n \sim D_1, \ldots, D_n$.
- $X_1, \ldots, X_n$ arrive in *adversarial order*. At step i, see realization u_i of X_i.
- Decide *immediately* and *irrevocably* to select or discard u_i.
- **Goal:** Select highest possible *single value*, compare against offline optimal $\mathbb{E}[\max_i X_i]$.

Prophet Inequality Problem



- Tight 1/2-competitive algorithm known [KS77].
- Set threshold $\tau = \frac{\mathbb{E}[\max_i X_i]}{2}$, accept first value above threshold [KW12].
- Applications: Posted Prices Mechanisms [HKS07; Cha+10].
- Related to *secretary problem* (no distributional information but random order).

Beyond Single Item ?

- Pick many elements, constraint on what you can pick.
- Additive functions:
  - Cardinality k: $1 - \frac{1}{\sqrt{k+3}}$ [Ala14]
  - Matroid: $\frac{1}{2}$ [KW12]
  - Matching: 0.337 [Ezr+20]
  - Downward-Closed: $O(\log n \cdot \log r)$ [Rub16]

Beyond Single Item ?

- Pick many elements, constraint on what you can pick.
- Additive functions:
  - Cardinality k: $1 - \frac{1}{\sqrt{k+3}}$ [Ala14]
  - Matroid: $\frac{1}{2}$ [KW12]
  - Matching: 0.337 [Ezr+20]
  - Downward-Closed: $O(\log n \cdot \log r)$ [Rub16]
- Several approaches. One idea: *Relaxation* + (online) *rounding*.
- Example: $X_i = \begin{cases} v_i & \text{w.p. } p_i \\ 0 & \text{otherwise} \end{cases}$

Solve $\max \sum_i x_i v_i$ then round x to get feasible S

$$x \in P(\mathcal{C})$$

$$\forall i \in N \quad x_i \leq p_i$$

Beyond Additive Functions ?

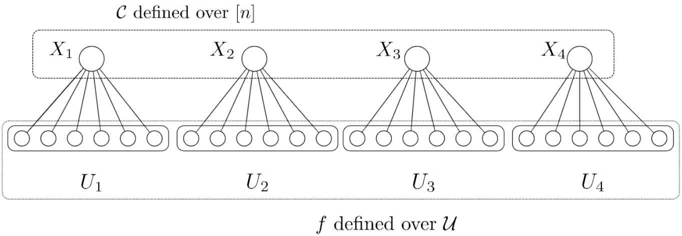
- *Submodular* functions:

$$\forall A, B \subseteq N : f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$$

  - If $A \subseteq B \implies f(A) \leq f(B)$, f is *monotone*.
  - Assume $f(S) \geq 0$ for all S.
- How to capture distributional information of PI model?

Submodular Prophet Inequality

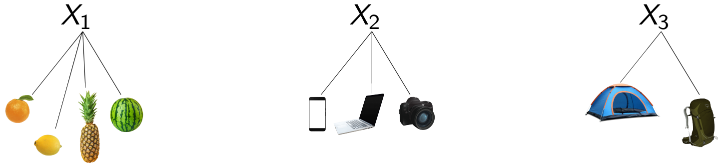
- Submodular Prophet Inequality (SPI):
  - Introduced by [RS16] to capture distributional information of X_i's.
  - X_i takes one value from U_i (correlated distribution), let $\mathcal{U} = \bigcup_{i=1}^{n} U_i$.
  - $f : 2^{\mathcal{U}} \to \mathbb{R}_{\geq 0}$, submodular.
  - Constraint $\mathcal{C}$ on N instead of $\mathcal{U}$.



- Model generalizes traditional prophet inequality problem.

Submodular Prophet Inequality

- Example:



Submodular Prophet Inequality

Theorem [RS16]

$\exists\, O(1)-$factor Submodular Prophet Inequalities for *single matroid* [RS16].

Submodular Prophet Inequality

Theorem [RS16]

$\exists\, O(1)$ −factor Submodular Prophet Inequalities for *single matroid* [RS16].

Proof Idea: Given marginals x^* of OPT,

1. Step 1:
   - *"Relax"* f into a continuous function, use *correlation gap*.

$$\begin{array}{c} \max f(S) \\ S \text{ is feasible} \end{array} \quad \Longrightarrow \quad \begin{array}{c} \max F(x) \\ x \in P \end{array}$$

   - Continuous Extensions:
     - Multilinear Extension: $F(x) = \mathbb{E}_{R\sim x}[f(R)]$.
     - Concave Closure: $f^+(x) = \max_{a_S} \left\{ \sum_S a_S f(S) \,\middle|\, a_S \geq 0, \sum_S a_S = 1, \forall i,\ \sum_{S:i\in S} a_S = x_i \right\}$.
   - Observe $f^+(x^*) \geq OPT$.

Submodular Prophet Inequality

Theorem [RS16]

$\exists\, O(1)$ −factor Submodular Prophet Inequalities for *single matroid* [RS16].

Proof Idea: Given marginals x^* of OPT,

1. Step 1:
   - *"Relax"* f into a continuous function, use *correlation gap*.

     $$\begin{array}{c} \max f(S) \\ S \text{ is feasible} \end{array} \implies \begin{array}{c} \max F(x) \\ x \in P \end{array}$$

   - Continuous Extensions:
     - Multilinear Extension: $F(x) = \mathbb{E}_{R \sim x}[f(R)]$.
     - Concave Closure:
       $f^+(x) = \max_{a_S} \left\{ \sum_S a_S f(S) \,\middle|\, a_S \geq 0, \sum_S a_S = 1, \forall i,\ \sum_{S: i \in S} a_S = x_i \right\}$.
   - Observe $f^+(x^*) \geq OPT$.
2. Step 2: *Round* x^* to get feasible S.

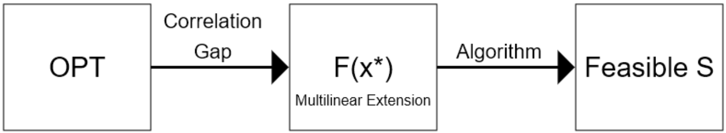
Correlation Gap

- $\inf_{x \in [0,1]^n} \frac{F(x)}{f^+(x)}$ is *correlation gap* of f.
- For *monotone submodular* f, correlation gap is at most $1 - \frac{1}{e}$ [Cal+11; Agr+12].
- For general submodular f (special case of subadditive), correlation gap can be arbitrarily large.
- To get around this, [RS16] define correlation gap *variant*:

$$\inf_{x \in [0,1]^n} \frac{F_{max}(x)}{f^+(x)} \geq \frac{1}{200}, \quad \text{where } F_{max}(x) = \max_{y \leq x} F(y).$$

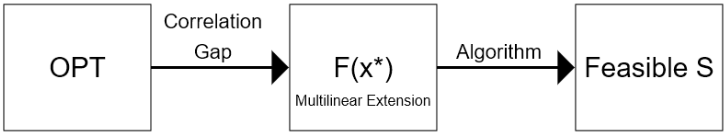
Rounding x^*

- [RS16] round x^* and get bound w.r.t. $F_{max}(x^*)$.



Rounding x^*

- [RS16] round x^* and get bound w.r.t. $F_{max}(x^*)$.



- $\exists$ rounding schemes which account for constraints: *Contention Resolution Scheme (CRS)* [CVZ11].
- *Online* CRS [FSZ16]: A CRS that rounds x in online manner.
- $\exists\, O(1)$-factor OCRSs for matroids, matchings, knapsacks, their intersection, etc [FSZ16].

Can we do better?

1. $\frac{1}{200}$ seems unreasonable for non-monotone correlation gap.
2. What about standard definition of correlation gap?
3. [RS16] technique only works for single matroid. What about more constraints [Luc17]?
4. Constant is in the thousands. Can we get improved bounds?
5. [RS16] assume knowledge of marginals x^* of OPT. Can we get results in polynomial time?

Results (1/3)

1. Via nice interpretation of known properties, we get $\inf_{x \in [0,1]^n} \frac{F_{max}(x)}{f^+(x)} \geq \frac{1}{e}$.

Results (1/3)

1. Via nice interpretation of known properties, we get $\inf_{x \in [0,1]^n} \frac{F_{max}(x)}{f^+(x)} \geq \frac{1}{e}$.
2. Parametrize by $p = \max_i x_i$. Use *Measured Continuous Greedy* algorithm to get

$$\frac{F(y)}{f^+(x)} \geq \begin{cases} 1 - \frac{1}{e} - p - \frac{\ln(1-p)}{e}, & p \in [0, 1 - 1/e] \\ 1/e, & p \in [1 - 1/e, 1] \end{cases}.$$

Results (2/3)

3. Standard definition of correlation gap:

Theorem

For any non-negative submodular $f : 2^N \to \mathbb{R}_{\geq 0}$ and any $x \in [0,1]^n$ s.t. $\max_i x_i \leq p$,

$$\frac{F(x)}{f^+(x)} \geq (1-p)\left(1 - \frac{1}{e}\right).$$

- Interesting on its own.

Results (3/3)

4. Rounding:

Theorem

For any constraint, there exists an *efficient black-box reduction* from Submodular Prophet Inequality to Online Contention Resolution Schemes.

- Generalizes rounding to several constraints.
- Allows for significantly improved bounds for SPI.

Results (3/3)

4. Rounding:

Theorem

For any constraint, there exists an *efficient black-box reduction* from Submodular Prophet Inequality to Online Contention Resolution Schemes.

- Generalizes rounding to several constraints.
- Allows for significantly improved bounds for SPI.

Feasibility constraint	Competitive Ratio	
	Monotone Submodular	General Submodular
Uniform matroid of rank $k \to \infty$	$1/4.3$	$1/17.2$
Matroid	$1/7.4$	$1/30$
Matching	$1/9.5$	$1/38$
Knapsack	$1/17.5$	$1/70$
Intersection of k matroids	$\Omega(1/k)$	$\Omega(1/k)$

Table 1: A summary of our results for several feasibility constraints.

Summary

1. *Generalize SPI framework* of [RS16] for arbitrary down-closed constraints for which we have an OCRS.
2. Fine-grained *correlation gap* for general non-negative submodular functions (w.r.t. $p = \max_i x_i$).
3. Fine-grained analysis of *Measured Continuous Greedy* algorithm.
4. Significantly *improved bounds for SPI* for several constraints in *polynomial time*.

Open Problems

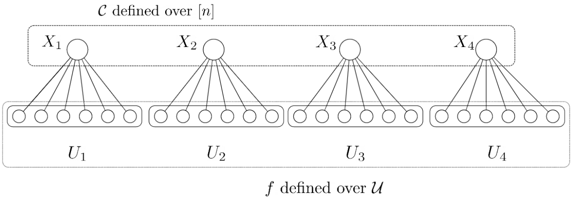
1. Can we get a tight $\frac{1}{2}-$submodular prophet inequality for a matroid constraint?
2. Can the $(1-p)\left(1-\frac{1}{e}\right)$-fine-grained correlation gap for general submodular functions be made tight?

THAT'S IT

THANK YOU!

QUESTIONS?

Rounding Subtleties



- Follows framework of [RS16].
- Recall: $\exists$ correlations! Exactly one element of U_i realizes at step i.
- *Main Idea:* Get rid of correlations by treating $X_i \sim Prod(U_i)$.
- Rubinstein-Singla's approach uses OCRS on $\mathcal{U} \implies$ only works on single matroid.
- Refined approach: Eliminate need for OCRS on $\mathcal{U}$, use OCRS only on $N \implies$ works for any constraint.

References I

Shipra Agrawal et al. "Price of Correlations in Stochastic Optimization". In: *Operations Research* 60.1 (2012), pp. 150–162. ISSN: 0030364X, 15265463. URL: http://www.jstor.org/stable/41476345.

Saeed Alaei. "Bayesian Combinatorial Auctions: Expanding Single Buyer Mechanisms to Many Buyers". In: *SIAM Journal on Computing* 43.2 (2014), pp. 930–972. DOI: 10.1137/120878422. eprint: https://doi.org/10.1137/120878422. URL: https://doi.org/10.1137/120878422.

References II

Gruia Calinescu et al. "Maximizing a Monotone Submodular Function Subject to a Matroid Constraint". In: *SIAM Journal on Computing* 40.6 (2011), pp. 1740–1766. DOI: 10.1137/080733991. eprint: https://doi.org/10.1137/080733991. URL: https://doi.org/10.1137/080733991.

Shuchi Chawla et al. "Multi-Parameter Mechanism Design and Sequential Posted Pricing". In: *Proceedings of the Forty-Second ACM Symposium on Theory of Computing*. STOC '10. Cambridge, Massachusetts, USA: Association for Computing Machinery, 2010, pp. 311–320. ISBN: 9781450300506. DOI: 10.1145/1806689.1806733. URL: https://doi.org/10.1145/1806689.1806733.

References III

Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. "Submodular Function Maximization via the Multilinear Relaxation and Contention Resolution Schemes". In: *Proceedings of the Forty-third Annual ACM Symposium on Theory of Computing*. STOC '11. San Jose, California, USA: ACM, 2011, pp. 783–792. ISBN: 978-1-4503-0691-1. DOI: 10.1145/1993636.1993740. URL: http://doi.acm.org/10.1145/1993636.1993740.

Tomer Ezra et al. "Online Stochastic Max-Weight Matching: Prophet Inequality for Vertex and Edge Arrival Models". In: *Proceedings of the 21st ACM Conference on Economics and Computation*. EC '20. Virtual Event, Hungary: Association for Computing Machinery, 2020, pp. 769–787. ISBN: 9781450379755. DOI: 10.1145/3391403.3399513. URL: https://doi.org/10.1145/3391403.3399513.

References IV

Moran Feldman, Ola Svensson, and Rico Zenklusen. "Online Contention Resolution Schemes". In: *Proceedings of the Twenty-seventh Annual ACM-SIAM Symposium on Discrete Algorithms*. SODA '16. Arlington, Virginia: Society for Industrial and Applied Mathematics, 2016, pp. 1014–1033. ISBN: 978-1-611974-33-1. URL: http://dl.acm.org/citation.cfm?id=2884435.2884507.

Mohammad Taghi Hajiaghayi, Robert Kleinberg, and Tuomas Sandholm. "Automated Online Mechanism Design and Prophet Inequalities". In: *Proceedings of the 22Nd National Conference on Artificial Intelligence - Volume 1*. AAAI'07. Vancouver, British Columbia, Canada: AAAI Press, 2007, pp. 58–65. ISBN: 978-1-57735-323-2. URL: http://dl.acm.org/citation.cfm?id=1619645.1619656.

References V

Ulrich Krengel and Louis Sucheston. "Semiamarts and finite values". In: *Bull. Amer. Math. Soc.* 83.4 (July 1977), pp. 745–747. URL: https://projecteuclid.org:443/euclid.bams/1183538915.

Robert Kleinberg and S. Matthew Weinberg. "Matroid Prophet Inequalities". In: *CoRR* abs/1201.4764 (2012). arXiv: 1201.4764. URL: http://arxiv.org/abs/1201.4764.

Brendan Lucier. "An Economic View of Prophet Inequalities". In: *SIGecom Exch.* 16.1 (Sept. 2017), pp. 24–47. DOI: 10.1145/3144722.3144725. URL: https://doi.org/10.1145/3144722.3144725.

Aviad Rubinstein and Sahil Singla. "Combinatorial Prophet Inequalities". In: *CoRR* abs/1611.00665 (2016). arXiv: 1611.00665. URL: http://arxiv.org/abs/1611.00665.

References VI

Aviad Rubinstein. "Beyond Matroids: Secretary Problem and Prophet Inequality with General Constraints". In: *Proceedings of the Forty-Eighth Annual ACM Symposium on Theory of Computing*. New York, NY, USA: Association for Computing Machinery, 2016, pp. 324–332. ISBN: 9781450341325. URL: https://doi.org/10.1145/2897518.2897540.