Combinatorial Optimization under Uncertainty

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2nd Drafting Workshop in Discrete Mathematics and Probability, Budapest, Hungary.

February 7th, 2023
A familiar* setup...

\[
\begin{align*}
\text{max} & \quad w^T x \\
\text{s.t.} & \quad x \in \mathcal{P} \\
& \quad x_i \in \{0, 1\} \quad \forall i
\end{align*}
\]
A familiar* setup...

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\begin{align*}
\text{max} & \quad \mathbf{w}^T \mathbf{x} \\
\text{s.t.} & \quad \mathbf{x} \in \mathcal{P} \quad \Rightarrow \quad \text{max} \quad \mathbf{w}^T \mathbf{x} \\
& \quad x_i \in \{0, 1\} \quad \forall i \\
& \quad 0 \leq x_i \leq 1 \quad \forall i
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\[
\Rightarrow
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& \quad x_i \in \{0, 1\} \quad \forall i \\
\end{align*}
\]
...with a twist!

Attempt #1
Create random set $R$ where $i \in R$ independently w.p. $x_i$ (*active elements*).

- $\mathbb{E}[\sum_{i \in R} w_i] = \sum_i w_i \cdot x_i$
- $R$ might be infeasible
...with a twist!

Attempt #1
Create random set $R$ where $i \in R$ independently w.p. $x_i$ (active elements).

- $\mathbb{E}\left[\sum_{i \in R} w_i\right] = \sum_{i} w_i \cdot x_i$
- $R$ might be infeasible

Attempt #2: Contention Resolution Scheme (CRS) $\pi$
[Chekuri, Vondrák and Zenklusen ’11]

1. Create random set $R$ where $i \in R$ independently w.p. $x_i$.
2. Drop elements from $R$ to create feasible $\pi(R)$. 
Contestation Resolution Schemes (CRSs)

c-selectability

CRS is $c$-selectable if
\[ \Pr [i \in \pi(R) \mid i \in R] \geq c \quad \forall i. \]

Theorem [Chekuri, Vondrák and Zenklusen ’11]

There exists a $(1 - 1/e)$-selectable CRS for matroid polytopes.

Holds if $R$ revealed in *uniformly random* order. What about adversarial order?
Online Contention Resolution Scheme (OCRS)
[Alaei '11, Feldman, Svensson and Zenklusen '15]

- \( \exists \frac{1}{2} \)-selectable OCRS for rank-1 matroids (tight). [Alaei '11]
- \( \exists \frac{1}{2} \)-selectable OCRS for matroids. [Lee, Singla '18]
Online Contention Resolution Schemes (OCRSs)

Online Contention Resolution Scheme (OCRS)
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![Diagram](image-url)

- **elements**
- **active elements**
- **selected elements**

∃1/2-selectable OCRS for rank-1 matroids (tight). [Alaei ’11]
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Online Contention Resolution Scheme (OCRS) [Alaei ’11, Feldman, Svensson and Zenklusen ’15]

Elements

Active elements

Selected elements

OCRS

∃1/2-selectable OCRS for rank-1 matroids (tight). [Alaei ’11]

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Why bother?

2. Rich connections to *Optimal Stopping Theory* – captures online decision making.
3. Rounding LPs online.
[Krengel, Sucheston and Garling '77]

\[ X_1, X_2, \ldots, X_n \overset{\text{ind.}}{\sim} (\text{known}) \ D_1, D_2, \ldots, D_n \]

arrive in \textit{adversarial} order.

- Design \textit{stopping time} to maximize selected value.
- Compare against all-knowing \textit{prophet}: \( \mathbb{E}[\max_i X_i] \).
Let’s play!

\[ u[2, 4] \] \[ u[2, 4] \] \[ u[1, 5] \] \[ u[0, 7] \]
Let’s play!

$X_1 = 3.91$

$\mathcal{U}[2, 4]$  
$\mathcal{U}[2, 4]$  
$\mathcal{U}[1, 5]$  
$\mathcal{U}[0, 7]$
Let’s play!

\( X_1 = 3.91 \)
\( X_2 = 3.56 \)
Let’s play!

$X_1 = 3.91$

$X_2 = 3.56$

$X_3 = 4.27$
Let’s play!

\[ U[2, 4] \]
\[ X_1 = 3.91 \]

\[ U[2, 4] \]
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\[ U[1, 5] \]
\[ X_3 = 4.27 \]

\[ U[0, 7] \]
\[ X_4 = 2.21 \]
Prophet Inequality [Krengel, Sucheston and Garling ’77, ’78]

∃ stopping strategy that achieves $\frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$, and this is tight.

\[ X_1 = 1 \text{ w.p. } 1, \text{ and } X_2 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases} \]

\[ \mathbb{E}[\text{ALG}] = 1 \text{ for all algorithms.} \]

\[ \mathbb{E}[\max_i X_i] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1 - \varepsilon) = 2 - \varepsilon. \]
Prophet Inequality [Krengel, Sucheston and Garling ’77, ’78]

∃ stopping strategy that achieves $\frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$, and this is tight.

$$X_1 = 1 \text{ w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$

$\mathbb{E}[\text{ALG}] = 1$ for all algorithms.

$$\mathbb{E}[\max_i X_i] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1 - \varepsilon) = 2 - \varepsilon.$$

- **Idea:** Set *threshold* $T$, accept first $X_i \geq T$.
  - $T : \Pr[\max_i X_i \geq T] = 1/2$ works [Samuel-Cahn ’84].
  - $T = \frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$ works [Kleinberg and Weinberg ’12].
Proof for rank-1 Matroids

\[ X^* = \max_i X_i \]
\[ p_i = \Pr[X^* = X_i] \implies \sum_i p_i = 1. \]

- \( \tau_i: \Pr[X_i \geq \tau_i] = p_i \)
- \( v_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)
Proof for rank-1 Matroids

\( X^* = \max_i X_i \)
\( p_i = \Pr[X^* = X_i] \implies \sum_i p_i = 1. \)

\( \tau_i: \Pr[X_i \geq \tau_i] = p_i \)
\( \nu_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)
\( \mathbb{E}[X^*] \leq \sum_i \nu_i(p_i) \cdot p_i, \)

since \( X^* \sim D^* \) with marginals \( p \).
Proof for rank-1 Matroids

Idea
Reject every random variable $X_i$ w.p. $1/2$.
Otherwise accept $i$ iff $X_i \geq \tau_i$ (happens w.p. $p_i$).

$$\mathbb{E}[ALG] = \sum_i \text{Pr}[\text{We reach } i] \cdot \frac{1}{2} \cdot p_i \cdot v_i(p_i)$$
Proof for rank-1 Matroids

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Reject every random variable $X_i$ w.p. $1/2$.
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By a union bound,
$$\Pr[\text{We reach } i] \geq \Pr[\text{We pick nothing}] \geq 1 - \sum_i \frac{p_i}{2} \geq \frac{1}{2}.$$  

$\triangleright$ 1/4-approximation to $\mathbb{E}[X^*]$.
Proof for rank-1 Matroids

Idea
Reject every random variable $X_i$ w.p. $1/2$.
Otherwise accept $i$ iff $X_i \geq \tau_i$ (happens w.p. $p_i$).

$$E[ALG] = \sum_i Pr[\text{We reach } i] \cdot \frac{1}{2} \cdot p_i \cdot v_i(p_i)$$

By a union bound,
$$Pr[\text{We reach } i] \geq Pr[\text{We pick nothing}] \geq 1 - \sum_i \frac{p_i}{2} \geq \frac{1}{2}.$$  

$\Rightarrow$ $1/4$-approximation to $E[X^*]$. 

Rewrite
$$E[ALG] \geq \sum_i r_i \cdot q_i \cdot p_i \cdot v_i(p_i)$$

Can we ensure $r_i \cdot q_i = 1/2$?

$\Rightarrow r_1 = 1 \Rightarrow q_1 = 1/2$. Then $r_{i+1} = r_i (1 - q_i p_i)$

$\Rightarrow$ If we set $q_i = \frac{1}{2r_i} \Rightarrow r_{i+1} = r_i - \frac{p_i}{2} = 1 - \sum_{j \leq i} \frac{p_i}{2} \geq \frac{1}{2}$
Proof for Rank-1 Matroids

Idea
Reject every random variable $X_i$ w.p. $1/2$.
Otherwise accept $i$ iff $X_i \geq \tau_i$ (happens w.p. $p_i$).

$$\mathbb{E}[ALG] = \sum_i \Pr[\text{We reach } i] \cdot 1/2 \cdot p_i \cdot v_i(p_i)$$

By a union bound,
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$\Rightarrow$ $1/4$-approximation to $\mathbb{E}[X^*]$.

Rewrite
$$\mathbb{E}[ALG] \geq \frac{1}{2} \cdot \sum_i x_i \cdot w_i$$
Adversaries

What does the adversary know?

 لهذا Offline: Nothing.
1/2-OCRS for rank-1 matroids, 1/2-OCRS for matroids.
Adversaries

What does the adversary know?

👍 **Offline:** Nothing.
- $\frac{1}{2}$-OCRS for rank-1 matroids, $\frac{1}{2}$-OCRS for matroids.

👍 **Online:** Same information as the Algorithm at every step.
- $\frac{1}{2}$-OCRS for rank-1 matroids, $\frac{1}{2}$-OCRS for matroids.
Adversaries

What does the adversary know?

- **Offline:** Nothing.
  - \(\frac{1}{2}\)-OCRS for rank-1 matroids, \(\frac{1}{2}\)-OCRS for matroids.

- **Online:** Same information as the Algorithm at every step.
  - \(\frac{1}{2}\)-OCRS for rank-1 matroids, \(\frac{1}{2}\)-OCRS for matroids.

- **Almighty:** All of \(R\) and randomness of the Algorithm.
  - \(\frac{1}{4}\)-OCRS for rank-1 matroids, \(\frac{1}{4}\)-OCRS for matroids.
What does the adversary know?

- **Offline:** Nothing.
  \(1/2\)-OCRS for rank-1 matroids, \(1/2\)-OCRS for matroids.

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  \(1/2\)-OCRS for rank-1 matroids, \(1/2\)-OCRS for matroids.

- **Almighty:** All of \(R\) and randomness of the Algorithm.
  \(1/4\)-OCRS for rank-1 matroids, \(1/4\)-OCRS for matroids.

**Greedy OCRS (Informal)**

Decides (randomly) which elements to select *before* it sees \(R\).

- Works against almighty adversary.
Adversaries

What does the adversary know?

👍 Offline: Nothing.
   1/2-OCRS for rank-1 matroids, 1/2-OCRS for matroids.

👍 Online: Same information as the Algorithm.
   1/2-OCRS for rank-1 matroids, 1/2-OCRS for matroids.

❓ Almighty: All of $R$ and randomness of the Algorithm.
   1/4-OCRS for rank-1 matroids, 1/4-OCRS for matroids.

Greedy OCRS (Formal)

Create $F_x \subseteq I$ before seeing $R$. When element $i$ arrives, greedily select $i$ iff $i \in R \ & \ S_{i-1} + i \in F_x$. 

Adversaries

What does the adversary know?

- **Offline:** Nothing.
  
  \( \frac{1}{2} \)-OCRS for rank-1 matroids, \( \frac{1}{2} \)-OCRS for matroids.

- **Online:** Same information as the Algorithm.
  
  \( \frac{1}{2} \)-OCRS for rank-1 matroids, \( \frac{1}{2} \)-OCRS for matroids.

- **Almighty:** All of \( R \) and randomness of the Algorithm.
  
  \( \frac{1}{4} \rightarrow \frac{1}{e} \)-OCRS for rank-1 matroids, \( \frac{1}{4} \)-OCRS for matroids.

**Greedy OCRS (Formal)**

Create \( \mathcal{F}_x \subseteq \mathcal{I} \) before seeing \( R \). When element \( i \) arrives, greedily select \( i \) iff \( i \in R \) & \( S_{i-1} + i \in \mathcal{F}_x \).

**Theorem [L. '22]**

\( \exists \frac{1}{e} \)-selectable Greedy OCRS for rank-1 matroids, and this is the best possible.
Recall $\mathbf{x}$ optimal solution to LP and $\sum_i x_i \leq 1$ (rank-1 matroid).
Recall $x$ optimal solution to LP and $\sum_i x_i \leq 1$ (rank-1 matroid).

Idea
Create set $\mathcal{F}_x$ where $i \in \mathcal{F}_x$ independently w.p. $\frac{1 - e^{-x_i}}{x_i}$.

Greedily select first $i \in R \cap \mathcal{F}_x$.

- Simulates ”splitting” $i$ into many small elements.
Proof

\[ \Pr[i \text{ is selected}] = \Pr[i \in F_x] \cdot \prod_{j < i} (1 - \Pr[j \text{ is selected}]) \]
\[ \geq \frac{1 - e^{-x_i}}{x_i} \prod_{j < i} \left(1 - x_j \cdot \frac{1 - e^{-x_j}}{x_j} \right) \]
\[ = \frac{1 - e^{-x_i}}{x_i} \prod_{j < i} e^{-x_j} \]
\[ = \frac{1 - e^{-x_i}}{x_i} \cdot e^{-\sum_{j < i} x_j} \]
\[ \geq \frac{(1 - e^{-x_i}) e^{x_i - 1}}{x_i} \]

(1) is minimized for \( x_i \to 0 \implies 1/e. \)

▶ Worst-case is \( n \to \infty \) and \( x_i \to 0 \ \forall i. \)

▶ Idea extends to partition and transversal matroids.
Other constraints & Open problems

1. $1/e$-Greedy OCRS for matroids?

2. $k$-Uniform Matroid:
   - ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei ’11]
   - ✓ $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-Greedy OCRS
Other constraints & Open problems

? $\frac{1}{e}$ -Greedy OCRS for matroids?

2. $k$-Uniform Matroid:
   ✓ $1 - O(\frac{1}{\sqrt{k}})$-OCRS [Alaei ’11]
   ✓ $1 - O\left(\sqrt{\log \frac{k}{k}}\right)$-Greedy OCRS

3. Matching:
   ? $\geq 0.349$-OCRS for bipartite, $\geq 0.344$-OCRS
      [MacRury, Ma, Grammel ’22]
   ? $\leq 0.433$-OCRS for bipartite, $\leq 0.4$-OCRS
      [MacRury, Ma, Grammel ’22]
   ? $\geq \frac{1}{2e}$ -Greedy OCRS [Feldman, Svensson, Zenklusen ’16]
   ? When $R$ is revealed in uniformly random order: $\leq \frac{1}{2}$-ROCRS
      [MacRury, Ma, Grammel ’22]
Other constraints & Open problems

1. $\frac{1}{e}$-Greedy OCRS for matroids?

2. $k$-Uniform Matroid:
   - $1 - O(\frac{1}{\sqrt{k}})$-OCRS [Alaei ’11]
   - $1 - O\left(\sqrt{\log \frac{k}{k}}\right)$-Greedy OCRS

3. Matching:
   - $\geq 0.349$-OCRS for bipartite, $\geq 0.344$-OCRS [MacRury, Ma, Grammel ’22]
   - $\leq 0.433$-OCRS for bipartite, $\leq 0.4$-OCRS [MacRury, Ma, Grammel ’22]
   - $\geq \frac{1}{2e}$-Greedy OCRS [Feldman, Svensson, Zenklusen ’16]
   - When $R$ is revealed in uniformly random order: $\leq \frac{1}{2}$-ROCRS [MacRury, Ma, Grammel ’22]

? Is $\frac{1}{2e}$-Greedy OCRS for matchings optimal?
Thank You!

Questions?