Combinatorial Optimization under Uncertainty

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A familiar* setup...

$$\begin{aligned} \max \quad & \boldsymbol{w}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in \mathcal{P} \\ & x_i \in \{0,1\} \quad \forall i \end{aligned}$$

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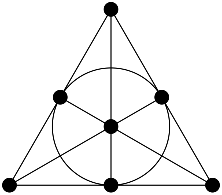

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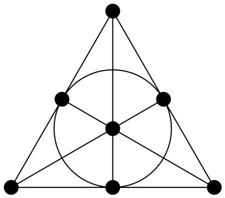

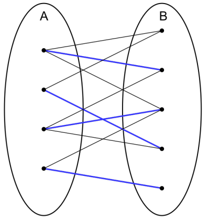

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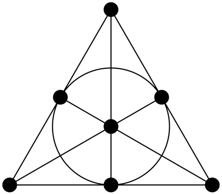

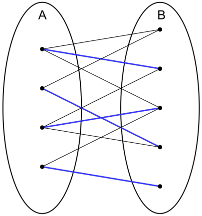

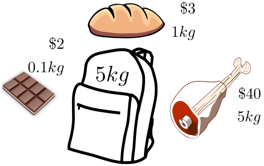

...with a twist!

Attempt #1

Create random set R where $i \in R$ independently w.p. x_i *(active elements)*.

- 👍 $\mathbb{E}[\sum_{i \in R} w_i] = \sum_i w_i \cdot x_i$
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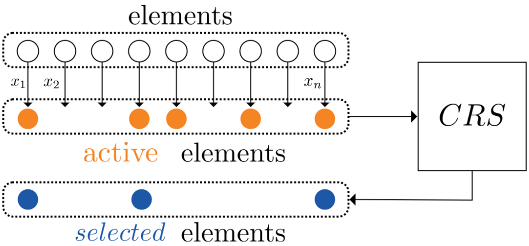
Attempt #2: Contention Resolution Scheme (CRS) π [Chekuri, Vondrák and Zenklusen '11]

1. Create random set R where $i \in R$ independently w.p. x_i.
2. Drop elements from R to create feasible $\pi(R)$.

Contention Resolution Schemes (CRSs)



c-selectability

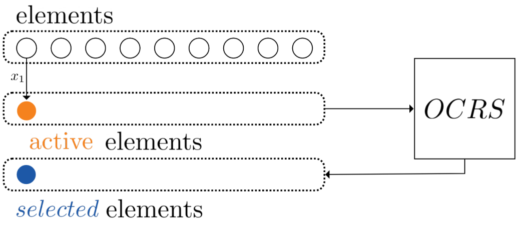
CRS is c-selectable if $\Pr[i \in \pi(R) \mid i \in R] \geq c \qquad \forall i.$

Theorem [Chekuri, Vondrák and Zenklusen '11]

There exists a $(1 - {}^1\!/_e)$-selectable CRS for matroid polytopes.

Holds if R revealed in *uniformly random* order. What about *adversarial* order?

Online Contention Resolution Scheme (OCRS)
[Alaei '11, Feldman, Svensson and Zenklusen '15]



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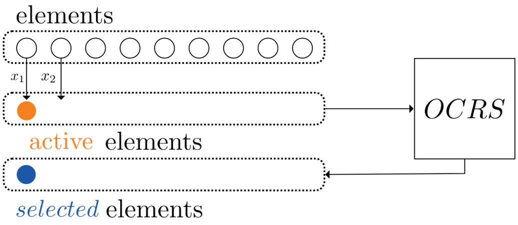

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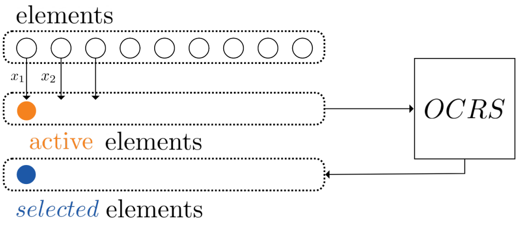

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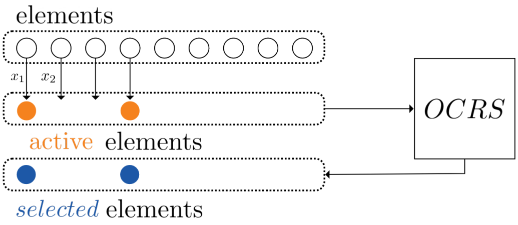

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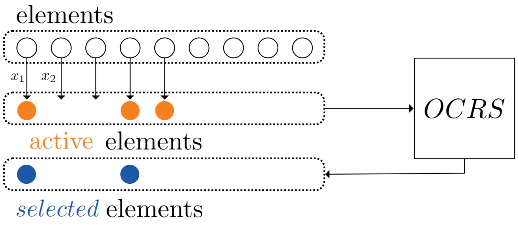

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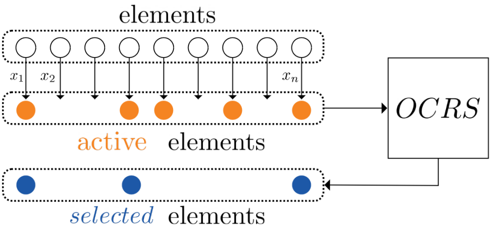

$\exists$ 1/2-selectable OCRS for rank-1 matroids (tight). [Alaei '11]
$\exists$ 1/2-selectable OCRS for matroids. [Lee, Singla '18]

Why bother?

1. Black-box composition for multiple constraints.
2. Rich connections to *Optimal Stopping Theory* – captures online decision making.
3. Rounding LPs online.

[Krengel, Sucheston and Garling '77]

$X_1, X_2, \ldots, X_n \overset{\text{ind.}}{\sim}$ (known) $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$
arrive in *adversarial* order.

- Design *stopping time* to maximize selected value.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.

Let's play!

$\mathcal{U}[2,4]$	$\mathcal{U}[2,4]$	$\mathcal{U}[1,5]$	$\mathcal{U}[0,7]$

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Prophet Inequality [Krengel, Sucheston and Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

$$X_1 = 1 \quad \text{w.p. } 1, \text{ and } X_2 = \begin{cases} 1/\varepsilon & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$$

$\mathbb{E}[\mathrm{ALG}] = 1$ for all algorithms.

$\mathbb{E}[\max_i X_i] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1 - \varepsilon) = 2 - \varepsilon$.

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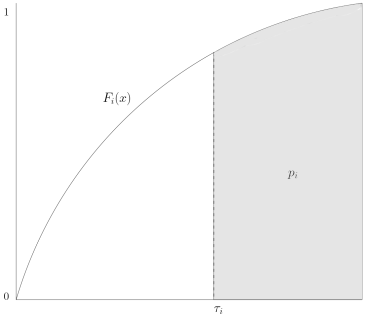
- Idea: Set *threshold* T, accept first $X_i \geq T$.
  - $T : \Pr[\max_i X_i \geq T] = 1/2$ works [Samuel-Cahn '84].
  - $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$ works [Kleinberg and Weinberg '12].

Proof for rank-1 Matroids

$X^* = \max_i X_i$
$p_i = \Pr[X^* = X_i] \implies \sum_i p_i = 1.$

- τ_i: $\Pr[X_i \geq \tau_i] = p_i$
- $v_i(p_i) := \mathbb{E}\left[X_i \mid X_i \geq \tau_i\right]$



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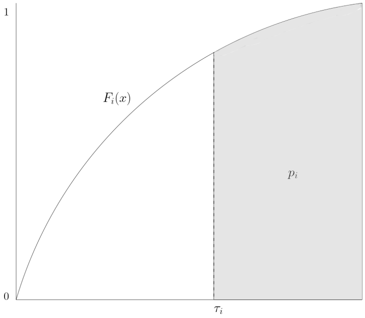
- $v_i(p_i) := \mathbb{E}\left[X_i \mid X_i \geq \tau_i\right]$
- $\mathbb{E}[X^*] \leq \sum_i v_i(p_i) \cdot p_i$,
  since $X^* \sim \mathcal{D}^*$ with marginals $\boldsymbol{p}$.



Idea

Reject every random variable X_i w.p. $1/2$.
Otherwise accept i iff $X_i \geq \tau_i$ (happens w.p. p_i).

$$\mathbb{E}[ALG] = \sum_i \Pr[\text{We reach } i] \cdot 1/2 \cdot p_i \cdot v_i(p_i)$$

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By a union bound,

$$\Pr[\text{We reach } i] \geq \Pr[\text{We pick nothing}] \geq 1 - \sum_i \frac{p_i}{2} \geq \frac{1}{2}.$$

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Rewrite

$$\mathbb{E}[ALG] \geq \sum_i r_i \cdot q_i \cdot p_i \cdot v_i(p_i)$$

Can we ensure $r_i \cdot q_i = 1/2$?

- $r_1 = 1 \implies q_1 = 1/2$. Then $r_{i+1} = r_i (1 - q_i p_i)$
- If we set $q_i = \frac{1}{2r_i} \implies r_{i+1} = r_i - \frac{p_i}{2} = 1 - \sum_{j \leq i} \frac{p_i}{2} \geq \frac{1}{2}$

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Rewrite

$$\mathbb{E}[ALG] \geq \frac{1}{2} \cdot \sum_i x_i \cdot w_i$$

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What does the adversary know?

- Offline: Nothing.
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Greedy OCRS (Informal)

Decides (randomly) which elements to select *before* it sees R.

- Works against almighty adversary.

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Greedy OCRS (Formal)

Create $\mathcal{F}_{\boldsymbol{x}} \subseteq \mathcal{I}$ before seeing R. When element i arrives, greedily select i iff $i \in R$ & $S_{i-1} + i \in \mathcal{F}_{\boldsymbol{x}}$.

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  $\cancel{1/4} \implies 1/e$ -OCRS for rank-1 matroids, $1/4$-OCRS for matroids.

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Theorem [L. '22]

$\exists\ 1/e$ -selectable Greedy OCRS for rank-1 matroids, and this is the best possible.

Proof

Recall $\boldsymbol{x}$ optimal solution to LP and $\sum_i x_i \leq 1$ (rank-1 matroid).

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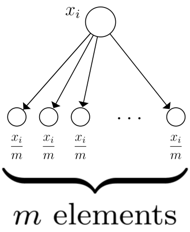
Idea

Create set $\mathcal{F}_{\boldsymbol{x}}$ where $i \in \mathcal{F}_{\boldsymbol{x}}$ independently w.p. $\frac{1-e^{-x_i}}{x_i}$.
Greedily select first $i \in R \cap \mathcal{F}_{\boldsymbol{x}}$.

- Simulates "splitting" i into many small elements.



Proof

$$
\begin{aligned}
\Pr[i \text{ is selected}] &= \Pr[i \in \mathcal{F}_{\boldsymbol{x}}] \cdot \prod_{j<i} (1 - \Pr[j \text{ is selected}]) \\
&\geq \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} \left(1 - x_j \cdot \frac{1 - e^{-x_j}}{x_j}\right) \\
&= \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} e^{-x_j} \\
&= \frac{1 - e^{-x_i}}{x_i} \cdot e^{-\sum_{j<i} x_j} \\
&\geq \frac{\left(1 - e^{-x_i}\right) e^{x_i - 1}}{x_i} \qquad (1)
\end{aligned}
$$

(1) is minimized for $x_i \to 0 \implies {}^1\!/_e$.

- Worst-case is $n \to \infty$ and $x_i \to 0 \ \forall i$.
- Idea extends to partition and transversal matroids.

Other constraints & Open problems

? $1/e$ -Greedy OCRS for matroids?

2. <u>k-Uniform Matroid:</u>
   - ✓ $1 - O\left(1/\sqrt{k}\right)$-OCRS [Alaei '11]
   - ✓ $1 - O\left(\sqrt{\log k / k}\right)$-Greedy OCRS

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   - ? $\geq 1/2e$ -Greedy OCRS [Feldman, Svensson, Zenklusen '16]
   - ? When R is revealed in uniformly random order: $\leq 1/2$-ROCRS [MacRury, Ma, Grammel '22]

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? Is $1/2e$ -Greedy OCRS for matchings optimal?

Questions?