Rounding LPs in a “Fair” Way: Many Questions, Few Answers

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Prophet Inequality

$X_1, X_2, \ldots, X_n \sim$ (known) $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n$ arrive in *adversarial* order.
Decide immediately and irrevocably to select or reject X_i.

- Design *stopping time* to maximize selected value.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.

Prophet Inequality [Krengel, Sucheston and Garling '77, '78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$,
and this is tight.

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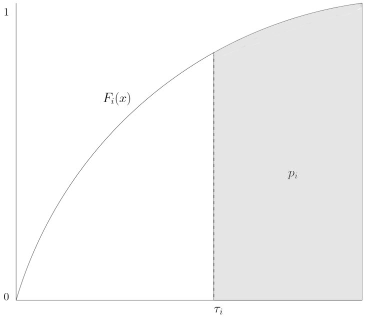
and this is tight.

- Idea: Set *threshold* T, accept first $X_i \geq T$.
  - $T : \Pr[\max_i X_i \geq T] = 1/2$ works [Samuel-Cahn '84].
  - $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$ works [Kleinberg and Weinberg '12].

Alternative Proof

$X^* = \max_i X_i$
$p_i = \Pr[X^* = X_i] \implies \sum_i p_i = 1.$

- τ_i: $\Pr[X_i \geq \tau_i] = p_i$
- $v_i(p_i) := \mathbb{E}\left[X_i \mid X_i \geq \tau_i\right]$



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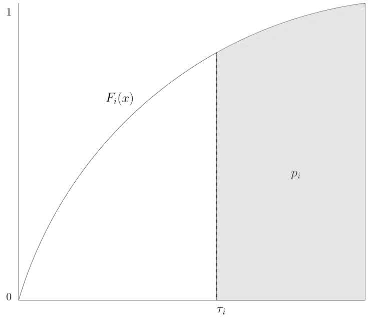
  since $X^* \sim \mathcal{D}^*$ with marginals $\boldsymbol{p}$.



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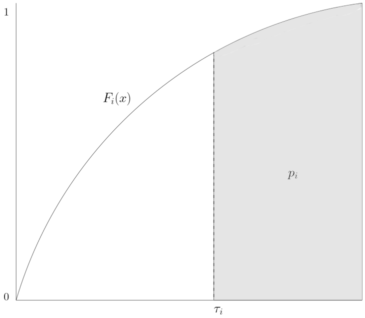
$$
\begin{aligned}
\max \quad & \sum_i v_i(z_i) \cdot z_i \\
\text{s.t.} \quad & \sum_i z_i \leq 1 \qquad\qquad (1) \\
& 0 \leq z_i \leq 1 \quad \forall i
\end{aligned}
$$



Idea

Reject every random variable X_i w.p. $1/2$.
Otherwise accept i iff $X_i \geq \tau_i$ (happens w.p. p_i).

$$\mathbb{E}[ALG] = \sum_i \Pr[\text{We reach } i] \cdot 1/2 \cdot p_i \cdot v_i(p_i)$$

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By a union bound,

$$\Pr[\text{We reach } i] \geq \Pr[\text{We pick nothing}] \geq 1 - \sum_i \frac{p_i}{2} \geq \frac{1}{2}.$$

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Rewrite

$$\mathbb{E}[ALG] \geq \sum_i r_i \cdot q_i \cdot p_i \cdot v_i(p_i)$$

Can we ensure $r_i \cdot q_i = 1/2$?

- $r_1 = 1 \implies q_1 = 1/2$. Then $r_{i+1} = r_i\,(1 - q_i p_i)$
- If we set $q_i = \frac{1}{2r_i} \implies r_{i+1} = r_i - \frac{p_i}{2} = 1 - \sum_{j \leq i} \frac{p_i}{2} \geq \frac{1}{2}$

How to generalize this?

$$\begin{aligned} \max \quad & \sum_i v_i(z_i) \cdot z_i \\ \text{s.t.} \quad & \sum_i z_i \leq 1 \\ & 0 \leq z_i \leq 1 \quad \forall i \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \max \quad & \sum_i w_i \cdot z_i \\ \text{s.t.} \quad & \boldsymbol{z} \in \mathcal{P}(\mathcal{M}) \\ & 0 \leq z_i \leq 1 \quad \forall i \end{aligned} \tag{2}$$

- $\boldsymbol{x}$: Optimal solution to (2). How to round $\boldsymbol{x}$?

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- $\boldsymbol{x}$: Optimal solution to (2). How to round $\boldsymbol{x}$?

Attempt #1

Create random set R where $i \in R$ independently w.p. x_i *(active elements)*.

- 👍 $\mathbb{E}[\sum_{i \in R} w_i] = \sum_i w_i \cdot x_i$
- 👎 R might be infeasible

How to generalize this?

Attempt #2: Contention Resolution Scheme (CRS) π

1. Create random set R where $i \in R$ independently w.p. x_i.
2. Drop elements from R to create feasible $\pi(R)$.

- [Chekuri, Vondrák and Zenklusen '11].

c-selectability

CRS is c-selectable if

$$\Pr\left[i \in \pi(R) \mid i \in R\right] \geq c \qquad \forall i.$$

- CRS is c-selectable $\implies$ c-approximation to LP.
- CRSs combine in black-box way for general constraints/objectives.

Contention Resolution Schemes (CRSs)

- Motivation: Submodular Function Maximization
- $f : 2^{\mathcal{N}} \rightarrow \mathbb{R}$ is submodular if
  $\forall A, B \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$.

Theorem [Chekuri, Vondrák and Zenklusen '11]

There exists a $(1 - {}^{1}/_{e})$-selectable CRS for matroid polytopes.

- Intuition: Greedy is optimal for matroids + a differential equality.

Correlation Gap

For constraint $\mathcal{C} = (\mathcal{N}, \mathcal{I})$, let $r_{\boldsymbol{w}}(S) = \max_{T \subseteq S, T \in \mathcal{I}} \sum_{i \in T} w_i$.

Correlation Gap

The *correlation gap* of $\mathcal{C}$ is defined as

$$\inf_{\boldsymbol{x} \in \mathcal{P}_{\mathcal{C}}, \boldsymbol{w} \geq 0} \frac{\mathbb{E}[r_{\boldsymbol{w}}(R(\boldsymbol{x}))]}{\sum_{i \in \mathcal{N}} w_i \, x_i},$$

where $i \in R$ independently w.p. x_i, $\forall i \in \mathcal{N}$.

Theorem [Chekuri, Vondrák and Zenklusen '11]

The correlation gap of the $r_{\boldsymbol{w}}$ for a constraint $\mathcal{C}$ is the same as the maximum c such that $\mathcal{C}$ admits a c-selectable CRS.

Proving Existence of CRSs via LPs? Very meta!

Fix $\boldsymbol{x}$, consider all mappings $\phi : 2^{\mathcal{N}} \to \mathcal{I}$.

$$\begin{array}{ll} \max & c \\ \text{s.t.} & \sum_{\phi} \lambda_{\phi} \Pr\left[i \in \phi(R(\boldsymbol{x})) \mid i \in R(\boldsymbol{x})\right] \geq c \quad \forall i \in \mathcal{N} \\ & \sum_{\phi} \lambda_{\phi} = 1 \\ & \lambda_{\phi} \geq 0 \quad \forall \phi \end{array}$$

Dual:

$$\begin{array}{ll} \min & \mu \\ \text{s.t.} & \sum_{i \in \mathcal{N}} z_i \Pr\left[i \in \phi(R(\boldsymbol{x})) \mid i \in R(\boldsymbol{x})\right] \leq \mu \quad \forall \phi \\ & \sum_{i \in \mathcal{N}} z_i = 1 \\ & z_i \geq 0 \quad \forall i \in \mathcal{N} \end{array}$$

Strong duality:

$$
\begin{aligned}
OPT &= \min_{\boldsymbol{z}} \max_{\phi} \sum_{i \in \mathcal{N}} z_i \Pr\left[i \in \phi(R(\boldsymbol{x})) \mid i \in R(\boldsymbol{x})\right] \\
&= \min_{\boldsymbol{z}} \max_{\phi} \sum_{i \in \mathcal{N}} z_i \frac{\Pr\left[i \in \phi(R(\boldsymbol{x}))\right]}{\Pr[i \in R(\boldsymbol{x})]}.
\end{aligned}
$$

Let $w_i = \frac{z_i}{x_i}$, recall that $x_i = \Pr[i \in R(\boldsymbol{x})]$.

$$
\begin{aligned}
OPT &= \min_{\boldsymbol{w}:\sum_i w_i x_i = 1} \max_{\phi} \sum_{i \in \mathcal{N}} w_i \Pr\left[i \in \phi(R(\boldsymbol{x}))\right] \\
&= \min_{\boldsymbol{w}:\sum_i w_i x_i = 1} \max_{\phi} \mathop{\mathbb{E}}_{S \leftarrow \phi(R(\boldsymbol{x}))} \left[\sum_{i \in S} w_i\right] \\
&= \min_{\boldsymbol{w}:\sum_i w_i x_i = 1} \mathbb{E}[r_{\boldsymbol{w}}(R(\boldsymbol{x}))] \\
&= \min_{\boldsymbol{w} \geq 0} \frac{\mathbb{E}[r_{\boldsymbol{w}}(R(\boldsymbol{x}))]}{\sum_{i \in \mathcal{N}} w_i x_i}.
\end{aligned}
$$

Open Problems

- $1 - {}^1/_e$ is tight for matroids [Yan '10].

Open Problem

What is the correlation gap for a matching constraint?

- ≥ 0.4326 [Bruggmann, Zenklusen '20], ≤ 0.544 [Karp, Sipser '81].
- ≥ 0.509 [Nuti, Vondrák '22], ≤ 0.544 [Karp, Sipser '81] for bipartite matching.
- More questions on knapsack constraints.

Online Contention Resolution Scheme (OCRS)

The elements of R (active elements) are revealed to the algorithm one by one in adversarial order.

- [Alaei '11, Feldman, Svensson and Zenklusen '15].
- $\exists\, 1/2$-selectable OCRS for single item (tight). $R = \{i \mid X_i \geq \tau_i\}$.
  Recall we guaranteed $\Pr[i \in \pi(R) \mid i \in R] = r_i \cdot q_i = 1/2$.

Why care about OCRSs?

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  Recall we guaranteed $\Pr[i \in \pi(R) \mid i \in R] = r_i \cdot q_i = {}^1\!/_2$.

Why care about OCRSs?

- c-selectable OCRS for $\mathcal{C} \implies c$-approximation to Prophet Inequality with constraint $\mathcal{C}$.
- c-approximation to (slight variant of) Prophet Inequality for $\mathcal{C} \implies c$-selectable OCRS for $\mathcal{C}$.
  Can use prophet inequalities to design optimal OCRSs!

Adversaries

What does the adversary know?

- Offline: Nothing.
  $1/2$-OCRS for single item, $1/2$-OCRS for matroids.

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Greedy OCRS (Informal)

Decides (randomly) which elements to select *before* it sees R.

- Works against almighty adversary.

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Greedy OCRS (Formal)

Create $\mathcal{F}_{\boldsymbol{x}} \subseteq \mathcal{I}$ before seeing R. When element i arrives, greedily select i iff $i \in R$ & $S_{i-1} + i \in \mathcal{F}_{\boldsymbol{x}}$.

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  ~~$1/4$~~ $\implies$ $1/e$ -OCRS for single item, $1/4$-OCRS for matroids.

Greedy OCRS (Formal)

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Theorem [L. '22]

$\exists 1/e$-selectable Greedy OCRS for single items, and this is the best possible.

Recall $\boldsymbol{x}$ optimal solution to LP and $\sum_i x_i \leq 1$ (single item).

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Idea

Create set T where $i \in T$ independently w.p. $\frac{1-e^{-x_i}}{x_i}$.
Greedily select i if $i \in R \cap T$.

- Simulates "splitting" i into many small elements.

$$
\begin{aligned}
\Pr[i \text{ is selected}] &= \Pr[i \in T] \cdot \prod_{j<i} (1 - \Pr[j \text{ is selected}]) \\
&\geq \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} \left(1 - x_j \cdot \frac{1 - e^{-x_j}}{x_j}\right) \\
&= \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} e^{-x_j} \\
&= \frac{1 - e^{-x_i}}{x_i} \cdot e^{-\sum_{j<i} x_j} \\
&\geq \frac{\left(1 - e^{-x_i}\right) e^{x_i - 1}}{x_i} \qquad (1)
\end{aligned}
$$

(1) is minimized for $x_i \to 0 \implies {}^1\!/_e$.

- Worst-case is $n \to \infty$ and $x_i \to 0 \;\forall i$.
- Idea extends to partition and transversal matroids.

More Open Problems!

1. *k-Uniform Matroid:*
   - ✓ $1 - O\left(1/\sqrt{k}\right)$-OCRS [Alaei '11]
   - ✓ $1 - O\left(\sqrt{\log k/k}\right)$-Greedy OCRS

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2. <u>General Matroid:</u>
   - ✓ $1/2$-OCRS [Lee, Singla '18]

Open Problem

Can we extend $1/e$-Greedy OCRS to general matroids?

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Can we extend $1/e$-Greedy OCRS to general matroids?

3. <u>Matching:</u>

Open Problem

What is the maximum c-OCRS for (bipartite / general) matchings?

- ≥ 0.349-OCRS for bipartite, ≥ 0.344-OCRS [MacRury, Ma, Grammel '22]
- ≤ 0.433-OCRS for bipartite, ≤ 0.4-OCRS [MacRury, Ma, Grammel '22]
- $\geq 1/2e \approx 0.184$-Greedy OCRS [Feldman, Svensson, Zenklusen '16]

Thank You!

Questions?