Rounding LPs in a “Fair” Way: Many Questions, Few Answers

Vasilis Livanos

livanos3@illinois.edu

University of Illinois Urbana-Champaign

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$X_1, X_2, \ldots, X_n \sim \text{(known)} \ D_1, D_2, \ldots, D_n$ arrive in adversarial order. Decide immediately and irrevocably to select or reject $X_i$.

- Design *stopping time* to maximize selected value.
- Compare against all-knowing *prophet*: $\mathbb{E}[\max_i X_i]$.

**Prophet Inequality** [Krengel, Sucheston and Garling ’77, ’78]

$\exists$ stopping strategy that achieves $\frac{1}{2} \cdot \mathbb{E}[\max_i X_i]$, and this is tight.
Prophet Inequality

\[X_1, X_2, \ldots, X_n \sim \text{(known)} \ \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n \text{ arrive in adversarial order.}\]

Decide immediately and irrevocably to select or reject \(X_i\).

- Design stopping time to maximize selected value.
- Compare against all-knowing prophet: \(E[\max_i X_i]\).

**Prophet Inequality** [Krengel, Sucheston and Garling ’77, ’78]

\(\exists\) stopping strategy that achieves \(1/2 \cdot E[\max_i X_i]\), and this is tight.

- **Idea:** Set threshold \(T\), accept first \(X_i \geq T\).
  - \(T : \Pr[\max_i X_i \geq T] = 1/2\) works [Samuel-Cahn ’84].
  - \(T = 1/2 \cdot E[\max_i X_i]\) works [Kleinberg and Weinberg ’12].
Alternative Proof

\[ X^* = \max_i X_i \]
\[ p_i = \Pr[X^* = X_i] \implies \sum_i p_i = 1. \]

- \( \tau_i \): \( \Pr[X_i \geq \tau_i] = p_i \)
- \( v_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)
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\( v_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)

\( \mathbb{E}[X^*] \leq \sum_i v_i(p_i) \cdot p_i, \)

since \( X^* \sim D^* \) with marginals \( p \).
Alternative Proof

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- \( \tau_i: \) \( \Pr[X_i \geq \tau_i] = p_i \)
- \( v_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)
- \( \mathbb{E}[X^*] \leq \sum_i v_i(p_i) \cdot p_i, \) since \( X^* \sim \mathcal{D}^* \) with marginals \( p. \)

\[
\max \sum_i v_i(z_i) \cdot z_i \\
s.t. \sum_i z_i \leq 1 \quad (1) \\
0 \leq z_i \leq 1 \quad \forall i
\]
Idea
Reject every random variable $X_i$ w.p. $1/2$. Otherwise accept $i$ iff $X_i \geq \tau_i$ (happens w.p. $p_i$).

$$\mathbb{E}[ALG] = \sum_i \Pr[\text{We reach } i] \cdot \frac{1}{2} \cdot p_i \cdot v_i(p_i)$$
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By a union bound,

$$\Pr[\text{We reach } i] \geq \Pr[\text{We pick nothing}] \geq 1 - \sum_i \frac{p_i}{2} \geq \frac{1}{2}.$$ 

$\triangleright$ $1/4$-approximation to $\mathbb{E}[X^*]$. Better?
**Idea**
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▶ $1/4$-approximation to $\mathbb{E}[X^*]$. Better?

Rewrite
$$\mathbb{E}[ALG] \geq \sum_i r_i \cdot q_i \cdot p_i \cdot v_i(p_i)$$

Can we ensure $r_i \cdot q_i = 1/2$?

▶ $r_1 = 1 \implies q_1 = \frac{1}{2}$. Then $r_{i+1} = r_i (1 - q_i p_i)$

▶ If we set $q_i = \frac{1}{2r_i} \implies r_{i+1} = r_i - \frac{p_i}{2} = 1 - \sum_{j \leq i} \frac{p_j}{2} \geq \frac{1}{2}$
How to generalize this?

\[
\begin{align*}
\text{max} & \quad \sum_i v_i(z_i) \cdot z_i \\
\text{s.t.} & \quad \sum_i z_i \leq 1, \quad 0 \leq z_i \leq 1 \quad \forall i
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad \sum_i w_i \cdot z_i \\
\text{s.t.} & \quad z \in \mathcal{P}(\mathcal{M}) \\
& \quad 0 \leq z_i \leq 1 \quad \forall i
\end{align*}
\]

\[\Rightarrow\] (2)

- x: Optimal solution to (2). How to round x?
How to generalize this?

\[
\max \sum_i v_i(z_i) \cdot z_i \\
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\max \sum_i w_i \cdot z_i \\
\text{s.t.} \quad z \in \mathcal{P}(\mathcal{M})
\]

\[0 \leq z_i \leq 1 \quad \forall i\]  \hspace{1cm} (2)

\[\triangleright x: \text{Optimal solution to (2). How to round } x?\]

**Attempt #1**

Create random set \( R \) where \( i \in R \) independently w.p. \( x_i \) (active elements).

\[\mathbb{E}[\sum_{i \in R} w_i] = \sum_i w_i \cdot x_i\]

\[\text{✓ } R \text{ might be infeasible}\]
How to generalize this?

Attempt #2: Contention Resolution Scheme (CRS) $\pi$

1. Create random set $R$ where $i \in R$ independently w.p. $x_i$.
2. Drop elements from $R$ to create feasible $\pi(R)$.

$\blacktriangleright$ [Chekuri, Vondrák and Zenklusen ’11].

$c$-selectability

CRS is $c$-selectable if

$$\Pr [i \in \pi(R) | i \in R] \geq c \quad \forall i.$$ 

$\blacktriangleright$ CRS is $c$-selectable $\implies$ $c$-approximation to LP.

$\blacktriangleright$ CRSs combine in black-box way for general constraints/objectives.
Motivation: Submodular Function Maximization

\[ f : 2^\mathcal{N} \rightarrow \mathbb{R} \text{ is submodular if} \]
\[ \forall A, B \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \]

**Theorem [Chekuri, Vondrák and Zenklusen '11]**
There exists a \((1 - \frac{1}{e})\)-selectable CRS for matroid polytopes.

Intuition: Greedy is optimal for matroids + a differential equality.
Correlation Gap

For constraint $C = (\mathcal{N}, \mathcal{I})$, let $r_w(S) = \max_{T \subseteq S, T \in \mathcal{I}} \sum_{i \in T} w_i$.

**Correlation Gap**

The *correlation gap* of $C$ is defined as

$$\inf_{x \in \mathcal{P}_C, w \geq 0} \frac{\mathbb{E}[r_w(R(x))]}{\sum_{i \in \mathcal{N}} w_i x_i},$$

where $i \in R$ independently w.p. $x_i$, $\forall i \in \mathcal{N}$.

**Theorem** [Chekuri, Vondrák and Zenklusen '11]

The correlation gap of the $r_w$ for a constraint $C$ is the same as the maximum $c$ such that $C$ admits a $c$-selectable CRS.
Proving Existence of CRSs via LPs? Very meta!

Fix $x$, consider all mappings $\phi : 2^\mathcal{N} \rightarrow \mathcal{I}$.

\[
\begin{align*}
\text{max} & \quad c \\
\text{s.t.} & \quad \sum_\phi \lambda_\phi \Pr \left[ i \in \phi(R(x)) \mid i \in R(x) \right] \geq c \quad \forall i \in \mathcal{N} \\
& \quad \sum_\phi \lambda_\phi = 1 \\
& \quad \lambda_\phi \geq 0 \quad \forall \phi
\end{align*}
\]

Dual:

\[
\begin{align*}
\text{min} & \quad \mu \\
\text{s.t.} & \quad \sum_{i \in \mathcal{N}} z_i \Pr \left[ i \in \phi(R(x)) \mid i \in R(x) \right] \leq \mu \quad \forall \phi \\
& \quad \sum_{i \in \mathcal{N}} z_i = 1 \\
& \quad z_i \geq 0 \quad \forall i \in \mathcal{N}
\end{align*}
\]
Strong duality:

\[
OPT = \min_z \max_\phi \sum_{i \in \mathcal{N}} z_i \Pr [i \in \phi(R(x)) | i \in R(x)]
\]

\[
= \min_z \max_\phi \sum_{i \in \mathcal{N}} z_i \frac{\Pr [i \in \phi(R(x))]}{\Pr[i \in R(x)]}.
\]

Let \( w_i = \frac{z_i}{x_i} \), recall that \( x_i = \Pr[i \in R(x)] \).

\[
OPT = \min_{w: \sum_i w_i x_i = 1} \max_\phi \sum_{i \in \mathcal{N}} w_i \Pr [i \in \phi(R(x))] \]

\[
= \min_{w: \sum_i w_i x_i = 1} \max_\phi \mathbb{E} \left[ S \leftarrow \phi(R(x)) \right] \left[ \sum_{i \in S} w_i \right] \]

\[
= \min_{w: \sum_i w_i x_i = 1} \mathbb{E}[r_w(R(x))] \]

\[
= \min_{w \geq 0} \frac{\mathbb{E}[r_w(R(x))]}{\sum_{i \in \mathcal{N}} w_i x_i}.
\]
Open Problems

1 \(- \frac{1}{e}\) is tight for matroids [Yan ’10].

Open Problem
What is the correlation gap for a matching constraint?

\[
\begin{align*}
\geq 0.4326 \text{ [Bruggmann, Zenklusen ’20], } & \leq 0.54 \text{ [Karp, Sipser ’81].} \\
\geq 0.4726 \text{ [Bruggmann, Zenklusen ’20], } & \leq 0.54 \text{ [Karp, Sipser ’81] for bipartite matching.}
\end{align*}
\]

More questions on knapsack constraints.
Online Contention Resolution Scheme (OCRS)

The elements of $R$ (active elements) are revealed to the algorithm one by one in adversarial order.

- [Alaei ’11, Feldman, Svensson and Zenklusen ’15].
- $\exists 1/2$-selectable OCRS for single item (tight). $R = \{ i \mid X_i \geq \tau_i \}$.

Recall we guaranteed $\Pr [i \in \pi(R) \mid i \in R] = r_i \cdot q_i = 1/2$.

Why care about OCRSs?
Online Contention Resolution Scheme (OCRS)

The elements of $R$ (active elements) are revealed to the algorithm one by one in adversarial order.

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Recall we guaranteed $\Pr[i \in \pi(R) \mid i \in R] = r_i \cdot q_i = 1/2$.

Why care about OCRSs?

- $c$-selectable OCRS for $C \implies c$-approximation to Prophet Inequality with constraint $C$.
- $c$-approximation to (slight variant of) Prophet Inequality for $C \implies c$-selectable OCRS for $C$.

Can use prophet inequalities to design optimal OCRSs!
Adversaries

What does the adversary know?

- **Offline:** Nothing.
  
  \( \frac{1}{2} \)-OCR for single item, \( \frac{1}{2} \)-OCR for matroids.

- **Online:** Same information as the Algorithm at every step.
  
  \( \frac{1}{2} \)-OCR for single item, \( \frac{1}{4} \)-OCR for matroids.

- **Almighty:** All realizations and randomness of the Algorithm.
  
  \( \frac{1}{4} \)-OCR for single item, \( \frac{1}{4} \)-OCR for matroids.

---

Greedy OCRS (Informal)

Decides (randomly) which elements to select before it sees \( R \).

Works against almighty adversary.
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  - 1/4-OCRS for single item, 1/4-OCRS for matroids.
Adversaries

What does the adversary know?

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**Greedy OCRS (Informal)**

Decides (randomly) which elements to select *before* it sees $R$.

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Adversaries

What does the adversary know?

- **Offline:** Nothing.
  1/2-OCR$S$ for single item, 1/2-OCR$S$ for matroids.

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  1/4-OCR$S$ for single item, 1/4-OCR$S$ for matroids.

**Greedy OCR$S$ (Formal)**

Create $\mathcal{F}_x \subseteq \mathcal{I}$ before seeing $R$. When element $i$ arrives, greedily select $i$ iff $i \in R \& S_{i-1} + i \in \mathcal{F}_x$.
Adversaries

What does the adversary know?

- **Offline:** Nothing.
  
  $1/2$-OCRS for single item, $1/2$-OCRS for matroids.

- **Online:** Same information as the Algorithm.
  
  $1/2$-OCRS for single item, $1/4$-OCRS for matroids.

- **Almighty:** All realizations and randomness of the Algorithm.
  
  $1/4 \implies 1/e$ -OCRS for single item, $1/4$-OCRS for matroids.

**Greedy OCRS (Formal)**

Create $F_x \subseteq I$ before seeing $R$. When element $i$ arrives, greedily select $i$ iff $i \in R$ & $S_{i-1} + i \in F_x$.

**Theorem [L. '22]**

$\exists 1/e$-selectable Greedy OCRS for single items, and this is the best possible.
Recall $x$ optimal solution to LP and $\sum_i x_i \leq 1$ (single item).
Recall $\mathbf{x}$ optimal solution to LP and $\sum_i x_i \leq 1$ (single item).

**Idea**
Create set $T$ where $i \in T$ independently w.p. $\frac{1-e^{-x_i}}{x_i}$. Greedily select $i$ if $i \in R \cap T$.

- Simulates "splitting" $i$ into many small elements.
\[ \text{Pr}[i \text{ is selected}] = \text{Pr}[i \in T] \cdot \prod_{j<i} (1 - \text{Pr}[j \text{ is selected}]) \]

\[ \geq \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} \left(1 - x_j \cdot \frac{1 - e^{-x_j}}{x_j} \right) \]

\[ = \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} e^{-x_j} \]

\[ = \frac{1 - e^{-x_i}}{x_i} \cdot e^{-\sum_{j<i} x_j} \]

\[ \geq (1 - e^{-x_i}) e^{x_i-1} \]

(1) is minimized for \( x_i \to 0 \implies \frac{1}{e}. \)

\[ \iff \text{Worst-case is } n \to \infty \text{ and } x_i \to 0 \ \forall i. \]

\[ \iff \text{Idea extends to partition and transversal matroids.} \]
More Open Problems!

1. k-Uniform Matroid:
   ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei ’11]
   ✓ $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-Greedy OCRS

2. General Matroid:
   ✓ $1/2$-OCRS [Lee, Singla ’18]

Open Problem
Can we extend $1/e$-Greedy OCRS to general matroids?

3. Matching:
   Open Problem
   What is the maximum $c$-OCRS for (bipartite / general) matchings?
   ▶ $\geq 0.349$-OCRS for bipartite, $\geq 0.344$-OCRS [MacRury, Ma, Grammel ’22]
   ▶ $\leq 0.433$-OCRS for bipartite, $\leq 0.4$-OCRS [MacRury, Ma, Grammel ’22]
   ▶ $\geq \frac{1}{2} e \approx 0.184$-Greedy OCRS [Feldman, Svensson, Zenklusen ’16]
More Open Problems!

1. \textit{k-Uniform Matroid:}
   \begin{itemize}
   \item ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei ’11]
   \item ✓ $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-Greedy OCRS
   \end{itemize}

2. \textit{General Matroid:}
   ✓ $\frac{1}{2}$-OCRS [Lee, Singla ’18]

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Can we extend $\frac{1}{e}$-Greedy OCRS to general matroids?
More Open Problems!

1. **k-Uniform Matroid:**
   - ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei ’11]
   - ✓ $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-Greedy OCRS

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- ≥ $0.349$-OCRS for bipartite, ≥ $0.344$-OCRS [MacRury, Ma, Grammel ’22]
- ≤ $0.433$-OCRS for bipartite, ≤ $0.4$-OCRS [MacRury, Ma, Grammel ’22]
- ≥ $\frac{1}{2}e \approx 0.184$-Greedy OCRS [Feldman, Svensson, Zenklusen ’16]
Thank You!

Questions?