Rounding LPs in a “Fair” Way: Many Questions, Few Answers

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Prophet Inequality

\(X_1, X_2, \ldots, X_n \sim \text{(known)} \ \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_n\) arrive in adversarial order. Decide immediately and irrevocably to select or reject \(X_i\).

▶ Design stopping time to maximize selected value.
▶ Compare against all-knowing prophet: \(\mathbb{E}[\max_i X_i]\).

Prophet Inequality [Krengel, Sucheston and Garling '77, '78]

\(\exists\) stopping strategy that achieves \(\frac{1}{2} \cdot \mathbb{E}[\max_i X_i]\), and this is tight.
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Prophet Inequality [Krengel, Sucheston and Garling ’77, ’78]

$\exists$ stopping strategy that achieves $1/2 \cdot \mathbb{E}[\max_i X_i]$, and this is tight.

- **Idea:** Set threshold $T$, accept first $X_i \geq T$.
  - $T : \Pr[\max_i X_i \geq T] = 1/2$ works [Samuel-Cahn ’84].
  - $T = 1/2 \cdot \mathbb{E}[\max_i X_i]$ works [Kleinberg and Weinberg ’12].
Alternative Proof

\[ X^* = \max_i X_i \]
\[ p_i = \Pr[X^* = X_i] \implies \sum_i p_i = 1. \]

- \( \tau_i \): \( \Pr[X_i \geq \tau_i] = p_i \)
- \( v_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)
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- \( v_i(p_i) := \mathbb{E}[X_i \mid X_i \geq \tau_i] \)
- \( \mathbb{E}[X^*] \leq \sum_i v_i(p_i) \cdot p_i, \)
  since \( X^* \sim \mathcal{D}^* \) with marginals \( p \).
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- \( \tau_i \): \( \Pr[X_i \geq \tau_i] = p_i \)
- \( v_i(p_i) := \mathbb{E}[X_i | X_i \geq \tau_i] \)
- \( \mathbb{E}[X^*] \leq \sum_i v_i(p_i) \cdot p_i, \) since \( X^* \sim D^* \) with marginals \( p \).

\[
\max \sum_i v_i(z_i) \cdot z_i \\
\text{s.t.} \quad \sum_i z_i \leq 1 \quad (1) \\
\quad 0 \leq z_i \leq 1 \quad \forall i
\]
Idea
Reject every random variable $X_i$ w.p. $1/2$.
Otherwise accept $i$ iff $X_i \geq \tau_i$ (happens w.p. $p_i$).

$$\mathbb{E}[ALG] = \sum_i \Pr[\text{We reach } i] \cdot \frac{1}{2} \cdot p_i \cdot v_i(p_i)$$
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$$

By a union bound,

$$
\Pr[\text{We reach } i] \geq \Pr[\text{We pick nothing}] \geq 1 - \sum_i \frac{p_i}{2} \geq \frac{1}{2}.
$$

$\blacksquare$ $1/4$-approximation to $\mathbb{E}[X^*]$. Better?
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By a union bound,
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$\downarrow$ 1/4-approximation to $\mathbb{E}[X^*]$. Better?

Rewrite
\[ \mathbb{E}[ALG] \geq \sum_i r_i \cdot q_i \cdot p_i \cdot v_i(p_i) \]

Can we ensure $r_i \cdot q_i = 1/2$?

$\downarrow$ $r_1 = 1 \implies q_1 = 1/2$. Then $r_{i+1} = r_i \left(1 - q_ip_i\right)$

$\downarrow$ If we set $q_i = \frac{1}{2r_i} \implies r_{i+1} = r_i - \frac{p_i}{2} = 1 - \sum_{j \leq i} \frac{p_j}{2} \geq \frac{1}{2}$
How to generalize this?

\[
\begin{align*}
\text{max} & \quad \sum_{i} v_i(z_i) \cdot z_i \\
\text{s.t.} & \quad \sum_{i} z_i \leq 1 \quad \Rightarrow \quad \text{s.t.} \quad z \in \mathcal{P}(\mathcal{M}) \\
& \quad 0 \leq z_i \leq 1 \quad \forall i \\
\end{align*}
\]

\[
\begin{align*}
\text{max} & \quad \sum_{i} w_i \cdot z_i \\
\text{s.t.} & \quad 0 \leq z_i \leq 1 \quad \forall i
\end{align*}
\]

▶ \( x \): Optimal solution to (2). How to round \( x \)?
How to generalize this?

$$\max \sum_i v_i(z_i) \cdot z_i \quad \text{s.t.} \quad \sum_i z_i \leq 1 \quad 0 \leq z_i \leq 1 \quad \forall i$$

$\implies$

$$\max \sum_i w_i \cdot z_i \quad \text{s.t.} \quad z \in P(M) \quad 0 \leq z_i \leq 1 \quad \forall i$$

$\blacktriangleright \quad x$: Optimal solution to (2). How to round $x$?

**Attempt #1**

Create random set $R$ where $i \in R$ independently w.p. $x_i$ (*active elements*).

- $\mathbb{E}[\sum_{i \in R} w_i] = \sum_i w_i \cdot x_i$
- $R$ might be infeasible
How to generalize this?

Attempt #2: Contention Resolution Scheme (CRS) $\pi$

1. Create random set $R$ where $i \in R$ independently w.p. $x_i$.
2. Drop elements from $R$ to create feasible $\pi(R)$.

- [Chekuri, Vondrák and Zenklusen ’11].

$c$-selectability

CRS is $c$-selectable if

$$\Pr [i \in \pi(R) \mid i \in R] \geq c \quad \forall i.$$ 

- CRS is $c$-selectable $\implies$ $c$-approximation to LP.
- CRSs combine in black-box way for general constraints/objectives.
Motivation: Submodular Function Maximization

$f : 2^\mathcal{N} \to \mathbb{R}$ is submodular if
\[ \forall A, B \quad f(A) + f(B) \geq f(A \cup B) + f(A \cap B). \]

**Theorem [Chekuri, Vondrák and Zenklusen ’11]**
There exists a $(1 - 1/e)$-selectable CRS for matroid polytopes.

**Intuition:** Greedy is optimal for matroids + a differential equality.
Correlation Gap

For constraint $C = (\mathcal{N}, \mathcal{I})$, let $r_w(S) = \max_{T \subseteq S, T \in \mathcal{I}} \sum_{i \in T} w_i$.

The correlation gap of $C$ is defined as

$$\inf_{x \in \mathcal{P}_C, w \geq 0} \frac{\mathbb{E}[r_w(R(x))]}{\sum_{i \in \mathcal{N}} w_i x_i},$$

where $i \in R$ independently w.p. $x_i$, $\forall i \in \mathcal{N}$.

Theorem [Chekuri, Vondrák and Zenklusen ’11]

The correlation gap of the $r_w$ for a constraint $C$ is the same as the maximum $c$ such that $C$ admits a $c$-selectable CRS.
Fix $\mathbf{x}$, consider all mappings $\phi : 2^\mathcal{N} \to \mathcal{I}$.

$$\begin{align*}
\text{max} \quad & c \\
\text{s.t.} \quad & \sum_{\phi} \lambda_\phi \Pr [i \in \phi(R(\mathbf{x})) \mid i \in R(\mathbf{x})] \geq c \quad \forall i \in \mathcal{N} \\
& \sum_{\phi} \lambda_\phi = 1 \\
& \lambda_\phi \geq 0 \quad \forall \phi
\end{align*}$$

Dual:

$$\begin{align*}
\text{min} \quad & \mu \\
\text{s.t.} \quad & \sum_{i \in \mathcal{N}} z_i \Pr [i \in \phi(R(\mathbf{x})) \mid i \in R(\mathbf{x})] \leq \mu \quad \forall \phi \\
& \sum_{i \in \mathcal{N}} z_i = 1 \\
& z_i \geq 0 \quad \forall i \in \mathcal{N}
\end{align*}$$
Strong duality:

\[ \text{OPT} = \min_z \max_{\phi} \sum_{i \in \mathcal{N}} z_i \Pr [i \in \phi(R(x)) \mid i \in R(x)] \]

\[ = \min_z \max_{\phi} \sum_{i \in \mathcal{N}} z_i \frac{\Pr [i \in \phi(R(x))]}{\Pr [i \in R(x)]}. \]

Let \( w_i = \frac{z_i}{x_i} \), recall that \( x_i = \Pr [i \in R(x)] \).

\[ \text{OPT} = \min_{w: \sum_i w_i x_i = 1} \max_{\phi} \sum_{i \in \mathcal{N}} w_i \Pr [i \in \phi(R(x))] \]

\[ = \min_{w: \sum_i w_i x_i = 1} \max_{\phi} \mathbb{E} \left[ \sum_{i \in S} w_i \right] \]

\[ = \min_{w: \sum_i w_i x_i = 1} \mathbb{E}[r_w(R(x))] \]

\[ = \min_{w \geq 0} \frac{\mathbb{E}[r_w(R(x))]}{\sum_{i \in \mathcal{N}} w_i x_i}. \]
1 − 1/e is tight for matroids [Yan ’10].

Open Problem

What is the correlation gap for a matching constraint?

- ≥ 0.4326 [Bruggmann, Zenklusen ’20], ≤ 0.544 [Karp, Sipser ’81].
- ≥ 0.509 [Nuti, Vondrák ’22], ≤ 0.544 [Karp, Sipser ’81] for bipartite matching.
- More questions on knapsack constraints.
Online Contention Resolution Scheme (OCRS)

The elements of $R$ (active elements) are revealed to the algorithm one by one in adversarial order.

- [Alaei '11, Feldman, Svensson and Zenklusen '15].
- $\exists 1/2$-selectable OCRS for single item (tight). $R = \{i \mid X_i \geq \tau_i\}$.

Recall we guaranteed $\Pr [i \in \pi(R) \mid i \in R] = r_i \cdot q_i = 1/2$.

Why care about OCRSs?
Online Contention Resolution Scheme (OCRS)

The elements of $R$ (active elements) are revealed to the algorithm one by one in adversarial order.

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Recall we guaranteed $\Pr[i \in \pi(R) \mid i \in R] = r_i \cdot q_i = 1/2$.

Why care about OCRSs?

- $c$-selectable OCRS for $C \implies c$-approximation to Prophet Inequality with constraint $C$.
- $c$-approximation to (slight variant of) Prophet Inequality for $C \implies c$-selectable OCRS for $C$.

Can use prophet inequalities to design optimal OCRSs!
What does the adversary know?

- **Offline**: Nothing.
  - ½-OCRS for single item, ½-OCRS for matroids.

- **Online**: Same information as the Algorithm at every step.
  - ½-OCRS for single item, ¼-OCRS for matroids.

- ** Almighty**: All realizations and randomness of the Algorithm.
  - ¼-OCRS for single item, ¼-OCRS for matroids.
Adversaries

What does the adversary know?

- **Offline**: Nothing.
  \(\frac{1}{2}\)-OCRS for single item, \(\frac{1}{2}\)-OCRS for matroids.

- **Online**: Same information as the Algorithm at every step.
  \(\frac{1}{2}\)-OCRS for single item, \(\frac{1}{4}\)-OCRS for matroids.
Adversaries

What does the adversary know?

- **Offline**: Nothing.  
  $\frac{1}{2}$-OCR for single item, $\frac{1}{2}$-OCR for matroids.

- **Online**: Same information as the Algorithm at every step.  
  $\frac{1}{2}$-OCR for single item, $\frac{1}{4}$-OCR for matroids.

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  $\frac{1}{4}$-OCR for single item, $\frac{1}{4}$-OCR for matroids.
Adversaries

What does the adversary know?

▶ Offline: Nothing.
  1/2-OCRS for single item, 1/2-OCRS for matroids.

▶ Online: Same information as the Algorithm at every step.
  1/2-OCRS for single item, 1/4-OCRS for matroids.

▶ Almighty: All realizations and randomness of the Algorithm.
  1/4-OCRS for single item, 1/4-OCRS for matroids.

Greedy OCRS (Informal)
Decides (randomly) which elements to select before it sees $R$.

▶ Works against almighty adversary.
Adversaries

What does the adversary know?

- **Offline:** Nothing.
  - 1/2-OCRS for single item, 1/2-OCRS for matroids.

- **Online:** Same information as the Algorithm.
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- **Almighty:** All realizations and randomness of the Algorithm.
  - 1/4-OCRS for single item, 1/4-OCRS for matroids.

**Greedy OCRS (Formal)**

Create $\mathcal{F}_x \subseteq \mathcal{I}$ before seeing $R$. When element $i$ arrives, greedily select $i$ iff $i \in R \& S_{i-1} + i \in \mathcal{F}_x$. 
Adversaries

What does the adversary know?

▶ **Offline**: Nothing.
   1/2-OCRS for single item, 1/2-OCRS for matroids.

▶ **Online**: Same information as the Algorithm.
   1/2-OCRS for single item, 1/4-OCRS for matroids.

▶ **Almighty**: All realizations and randomness of the Algorithm.
   1/4 $\implies$ 1/e-OCRS for single item, 1/4-OCRS for matroids.

**Greedy OCRS (Formal)**

Create $\mathcal{F}_x \subseteq \mathcal{I}$ before seeing $R$. When element $i$ arrives, greedily select $i$ iff $i \in R$ & $S_{i-1} + i \in \mathcal{F}_x$.

**Theorem [L. ’22]**

$\exists$ 1/e-selectable Greedy OCRS for single items, and this is the best possible.
Recall $x$ optimal solution to LP and $\sum_i x_i \leq 1$ (single item).
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Idea
Create set $T$ where $i \in T$ independently w.p. $1 - e^{-x_i} x_i$.
Greedily select $i$ if $i \in R \cap T$.

- Simulates ”splitting” $i$ into many small elements.
Pr[i is selected] = Pr[i ∈ T] · \( \prod_{j<i} (1 - \Pr[j \text{ is selected}]) \)

\[
\geq \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} \left( 1 - x_j \cdot \frac{1 - e^{-x_j}}{x_j} \right)
\]

\[
= \frac{1 - e^{-x_i}}{x_i} \prod_{j<i} e^{-x_j}
\]

\[
= \frac{1 - e^{-x_i}}{x_i} \cdot e^{-\sum_{j<i} x_j}
\]

\[
\geq \frac{(1 - e^{-x_i}) e^{x_i-1}}{x_i}
\]

(1) is minimized for \( x_i \to 0 \implies 1/e. \)

- Worst-case is \( n \to \infty \) and \( x_i \to 0 \ \forall i. \)

- Idea extends to partition and transversal matroids.
More Open Problems!

1. $k$-Uniform Matroid:
   ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei ‘11]
   ✓ $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-Greedy OCRS

2. General Matroid:
   ✓ $\frac{1}{2}$-OCRS [Lee, Singla ‘18]

Open Problem
Can we extend $\frac{1}{e}$-Greedy OCRS to general matroids?

3. Matching:
   Open Problem
   What is the maximum $c$-OCRS for (bipartite / general) matchings?
   
   ▶ $\geq \frac{349}{344}$-OCRS for bipartite,
   
   ▶ $\leq \frac{433}{4}$-OCRS [MacRury, Ma, Grammel ‘22]

   ▶ $\geq \frac{1}{2}e \approx 0.184$-Greedy OCRS [Feldman, Svensson, Zenklusen ‘16]
More Open Problems!

1. *k*-Uniform Matroid:
   - ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei '11]
   - ✓ $1 - O\left(\sqrt{\log\frac{k}{k}}\right)$-Greedy OCRS

2. General Matroid:
   - ✓ $1/2$-OCRS [Lee, Singla '18]

Open Problem

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More Open Problems!

1. **k-Uniform Matroid:**
   - ✓ $1 - O\left(\frac{1}{\sqrt{k}}\right)$-OCRS [Alaei ’11]
   - ✓ $1 - O\left(\sqrt{\frac{\log k}{k}}\right)$-Greedy OCRS

2. **General Matroid:**
   - ✓ $\frac{1}{2}$-OCRS [Lee, Singla ’18]

**Open Problem**

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3. **Matching:**

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What is the maximum $c$-OCRS for (bipartite / general) matchings?

- ≥ $0.349$-OCRS for bipartite, ≥ $0.344$-OCRS [MacRury, Ma, Grammel ’22]
- ≤ $0.433$-OCRS for bipartite, ≤ $0.4$-OCRS [MacRury, Ma, Grammel ’22]
- ≥ $\frac{1}{2e} \approx 0.184$-Greedy OCRS [Feldman, Svensson, Zenklusen ’16]
Thank You!

Questions?