Simple & Optimal Greedy Online Contention Resolution Schemes
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Model

- **Input:**
  - Linear or Concave Program:
    \[
    \max_c \sum_i c_i z_i \\
    \text{s.t. } z_i \in P(C), 0 \leq z_i \leq 1
    \]
  - or
    \[
    \max_i g(z_i) \\
    \text{s.t. } z_i \in P(C), 0 \leq z_i \leq 1
    \]
  - Point \( x \in \text{polytope } P(C) \).
- **Goal:**
  - Round \( x \) while respecting constraint \( C \).
- **Idea:**
  - Draw random \( R \sim \text{Prod}(x) \) (active elements), i.e., \( i \in R \) independently with probability \( x_i \) \( \forall i \).
  - \( R \) might be infeasible.
- **Idea # 2:**
  - "Drop" elements to obtain feasible \( \pi(R) \subseteq R \).
- **Output:**
  - Algorithm \( \pi : 2^N \to 2^N \) s.t.
    \[
    \Pr[i \in \pi(R) | i \in R] \geq \alpha
    \]
  - for largest \( \alpha \) possible \( \Rightarrow \alpha\)-approximation.

Background

Algorithm \( \pi \) is called an **Online Contention Resolution Scheme (OCRS)**. To compare against almighty adversary, we need:

**Definition 1.**
- A **Greedy OCRS** \( \pi \):
  - Defines a subfamily of feasible sets \( F_{\pi,x} \subseteq C \).
  - When \( i \in R \) arrives, \( \pi \) selects \( i \) if \( \pi(R_{i-1}) + i \in F_{\pi,x} \).

**Theorem 1**

There exists a \( 1/e \)-selectable greedy OCRS for
- rank-1 matroids.
- partition matroids.
- transversal matroids.

**Theorem 2**

No greedy OCRS can be \( (1/e + \varepsilon) \)-selectable, even for rank-1 matroids.

Algorithm

- **Main Idea:**
  - Create set \( T \subseteq N \), where \( i \in T \) independently with probability \( 1 - \varepsilon \).
  - i.e., \( F_{\pi,x} = \{i\} | i \in T \).
  - When \( i \) comes, greedily accept it if \( i \in R \cap T \).
- **Intuition:**
  - Simulates splitting \( i \) into \( m \) elements with probability \( x_i / m \) each, as \( m \to \infty \).
- **Generalization:**
  - Doesn’t work for transversal matroids \( \Rightarrow \) need different \( \pi \).
  - Better \( (1 - 1/e) \)-approximation when \( |N(u)| \geq 3 \) for all elements \( u \).

Experiments

N = 5 to 100, iterations = 200,000, repetitions = 10