Prophet Inequalities for Cost Minimization

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Motivation

- Want to sell an orange. We see $n$ buyers \textit{sequentially}.
- Buyer $i$ has private valuation $v_i$. How to offer prices?
  - Option 1: Run an auction. Meh.
Motivation

- Want to sell an orange. We see $n$ buyers *sequentially*.
- Buyer $i$ has private valuation $v_i$. How to offer prices?
  - Option 1: Run an auction. Meh.
  - Option 2: Become a grocer!
- Plan:
  1. Set price $T$.
  2. Leave store.
  3. ???
  4. Profit.
Prophet Inequality

- Worst-case order + unknown $v_i$'s = Can't do anything.
- Random order + unknown $v_i$'s = Secretary problem.
- Worst-case order + some knowledge of $v_i$'s = Prophet inequality.
Prophet Inequality

- Worst-case order + unknown \( v_i \)'s = Can't do anything.
- Random order + unknown \( v_i \)'s = Secretary problem.
- Worst-case order + some knowledge of \( v_i \)'s = Prophet inequality.
- \( n \) random variables \( X_1, \ldots, X_n \sim D_1, \ldots, D_n \) arriving in adversarial order.
- Step \( i \) \( \implies \) observe realization \( x_i \).
  - Accept \( x_i \) \( \implies \) Game ends.
  - Reject \( x_i \) \( \implies \) Step \( i + 1 \).
- **Goal**: Select \( \max_i x_i \).
Let’s Play

\[
\begin{align*}
\end{align*}
\]
Let’s Play

\[ x_1 = 2.74 \]
Let’s Play

- $x_1 = 2.74$
- $x_2 = 3.75$
Let’s Play

\begin{align*}
U[2, 4] & \quad U[2, 4] & \quad U[1, 5] & \quad U[0, 7] \\
\bullet \ x_1 &= 2.74 \\
\bullet \ x_2 &= 3.75 \\
\bullet \ x_3 &= 2.81
\end{align*}
Let’s Play

\[ x_1 = 2.74 \]
\[ x_2 = 3.75 \]
\[ x_3 = 2.81 \]
\[ x_4 = 5.66 \]
Prophet Inequality

- ∃ algorithm $\mathcal{A}$ s.t. $\mathbb{E}[\mathcal{A}] \geq \frac{1}{2} \mathbb{E} [\max_{i=1}^{n} X_i]$, and this is tight [KS77].
- Many algorithms achieve $1/2$.

$\mathcal{A}_T$: “Fixed-Threshold” Algorithm

Set threshold $T$ based on $D_1, \ldots, D_n$. Accept first $x_i \geq T$.

- How to choose $T$?
Prophet Inequality

- \exists \text{ algorithm } A \text{ s.t. } \mathbb{E}[A] \geq \frac{1}{2} \mathbb{E}[\max_{i=1}^{n} X_i], \text{ and this is tight } [KS77].
- Many algorithms achieve 1/2.

\( A_T \): “Fixed-Threshold” Algorithm

Set threshold \( T \) based on \( D_1, \ldots, D_n \). Accept first \( x_i \geq T \).

- How to choose \( T \)?
  1. Set \( T = \text{median of the distribution of } \max_{i=1}^{n} X_i \), i.e. \( \Pr[\max_{i=1}^{n} X_i \geq T] = \frac{1}{2} \) [Sam84].
  2. Set \( T = \frac{1}{2} \mathbb{E}[\max_{i=1}^{n} X_i] \) [KW12].
$X_1 = 1 \text{ w.p. } 1$, and $X_2 = \begin{cases} \frac{1}{\varepsilon} & \text{w.p. } \varepsilon \\ 0 & \text{w.p. } 1 - \varepsilon \end{cases}$.

For every algorithm $\mathcal{A}$, $\mathbb{E}[\mathcal{A}] = 1$.

Prophet:

$$\mathbb{E}[\max_i X_i] = \frac{1}{\varepsilon} \cdot \varepsilon + 1 \cdot (1 - \varepsilon) = 2 - \varepsilon.$$
If we are buying? Same problem?

Cost Prophet Inequality: Select $\min_i x_i$ subject to selecting at least one $i$. 
Cost Prophet Inequality

- If we are buying? Same problem?
- Cost Prophet Inequality: Select \( \min_i x_i \) subject to selecting at least one \( i \).

\[
X_1 = 1 \text{ w.p. 1, and } X_2 = \begin{cases} 0 & \text{w.p. } 1 - 1/L \\ L & \text{w.p. } 1/L \end{cases}.
\]

For every algorithm \( \mathcal{A} \), \( \mathbb{E}[\mathcal{A}] = 1 \).

Prophet:

\[
\mathbb{E}[\min_i X_i] = 1 \cdot \frac{1}{L} + 0 \cdot \left(1 - \frac{1}{L}\right) = \frac{1}{L}.
\]
What *can* we do?

- Focus on I.I.D. $X_1, \ldots, X_n \sim D$.
- Single threshold not good enough*, need multiple thresholds.

**Theorem 1: Optimal Threshold Algorithm $\mathcal{A}$**

Let $G(i)$ be $\mathcal{A}$’s expected value, when it sees $X_i, \ldots, X_n$.

The algorithm $\mathcal{A}$ which sets $\tau_i = G(i + 1)$ and accepts the first $i$ such that $X_i \leq \tau_i$ is optimal.
Detour: Hazard Rate

- Need a tool to classify different distributions.

**Hazard Rate (aka Failure Rate)**

For a distribution $D$ with cdf $F$ and pdf $f$, the **hazard rate** is defined as

$$h(x) \triangleq \frac{f(x)}{1 - F(x)}.$$

- Intuition: $h(x) = \Pr [X = x \mid X \geq x]$ (for discrete distributions).
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Hazard Rate (aka Failure Rate)

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Intuition: $h(x) = \Pr [X = x \mid X \geq x]$ (for discrete distributions).

$h$ monotonically increasing $\implies$ Monotone Hazard Rate (MHR) distribution.

MHR distributions don’t have heavy tails.

Important in revenue maximization via virtual valuations.
Cumulative Hazard Rate: $H(x) = \int_0^x h(u) \, du$.

**Idea:** Study distributions with polynomial $H$; let’s call them $P_H$. Approximate all* other distributions via polynomials.

$$H(x) = \sum_{i=1}^{k} a_i x^{d_i}, \quad 0 < d_1 \leq \cdots \leq d_k.$$ 

$d_1$ controls how heavy $D$’s tail is.

$d_1 \geq 1 \implies$ MHR distribution.
Theorem 2

For every distribution $D \in P_H$ and I.I.D. random variables drawn from $D$, there exists a $\lambda(d_1)$-competitive cost prophet inequality, where

$$\lambda(d_1) = \frac{(1 + 1/d_1)^{1/d_1}}{\Gamma(1 + 1/d_1)}.$$

Furthermore, this is tight for $H(x) = x^{d_1}$.

- Via Stirling’s approximation

$$\lambda(d_1) \approx c \cdot e^{1/d_1}.$$
Both! Ratio can be arbitrarily bad, but constant for every fixed $D$. 
Special Case: MHR Distributions

- For $d_1 \geq 1 \Rightarrow$ distribution is MHR.
- Special case of “regular” distributions; exponential-like.
- $\lambda(d_1)$ decreasing in $d_1 \Rightarrow$

$$\lambda(d_1) \leq \frac{(1 + 1/1)^1}{\Gamma(1 + 1/1)} = 2,$$

for all MHR distributions.
- 2 is tight for the exponential distribution.
Single Threshold

- Single threshold suffices for single-item classical prophet inequality!
- Impossible for cost prophet inequality.

**Theorem 3**

For every distribution $D \in P_H$ and I.I.D. random variables drawn from $D$, there exists a single threshold $T$ such that accepting the first $i$ where $X_i \leq T$ yields an $O\left((\log n)^{1/d_1}\right)$-approximation to $\mathbb{E}[\min_i X_i]$. Furthermore, this is tight for $H(x) = x^{d_1}$. 
1. Optimal-threshold algorithm characterization for CPI.
2. Distribution-dependent constant for polynomial $H$.
3. Universal constant 2 for MHR distributions.
Open Questions

- Only use $k$-thresholds for $1 < k < n$. How does the ratio change?
- Get universal constant for subclass of distributions, like MHR; maybe \textit{regular}?
- Only have sample access to $\mathcal{D}$, how does the ratio with the number of samples?
- Impossibility result does not apply in the \textit{free order} setting. I.I.D. case is upper bound, but is a distribution-dependent constant-factor ratio possible?
QUESTIONS ?
