Model

- $n$ agents: $N$.
- $m$ indivisible items: $M$.
- Valuations $v_i(S)$, $\forall$ agents $i$ and sets of items $S$.
- Goal: Find allocation $A$ that is
  - Fair: EFX and PropMX.
  - Efficient: Pareto-Optimality (PO).
- Challenge: Items are mixed manna, i.e. item $j$ might be a good for some agent $i$ ($v_{ij} \geq 0$) and a bad for another agent $i'$ ($v_{ij} < 0$).
- Types of Instances:
  - Separable: $M = M^+ \cup M^-$, where $M^+$ are goods for all and $M^-$ are bads for all.
  - Restricted Mixed Goods (RMG): $\forall j \in M$, either $v_{ij} \leq 0$ or $v_{ij} = v_j$ for all agents $i$.

Results

- All allocations are efficiently computable.
- Separable instances: PropMX allocation.
- RMG instances:
  - PropMX allocation for general pure bads.
  - EFX + PropMX allocation for IDO pure bads.
  - EFX + PropMX + PO allocation for identical pure bads.

Background

- An allocation $A$ is:
  - EFX: $\forall i, j \in N$, either we have that $\forall j \in A_i$ where $v_{ij} > 0$, $v_j(A) - v_j \geq v_j(A_i)$, or we have that $\forall j \in A_i$ where $v_{ij} < 0$, $v_j(A) - v_j \geq v_j(A_i)$.
  - PropMX: $\forall i \in N$, either we have $v_j(A) + \max_{j' \in A_i \setminus \{j\}} v_{ij'} \geq \frac{1}{n} \cdot v(M)$, or $\forall j \in A_i$ where $v_{ij} < 0$, we have $v_j(A) - v_j \geq \frac{1}{n} \cdot v(M)$.
  - PO: If $\exists$ allocation $A'$ such that $v_j(A') > v_j(A_i)$ for all $i \in N$ and $v_j(A_i) > v_j(A_i)$ for at least one $i \in N$.
- Pure bads: Set $M^-$ of items such that $v_j < 0$ for all $i \in N$ and $j \in M^-$.
- $M^-$ is IDO if: $\exists$ ordering of $M^-$ such that $\forall i \in N$, $v_i \leq v_2 \leq \cdots \leq v_{|M^-|}$.
- $M^-$ is identical if: $\forall j \in M^-$, $\forall v_j < 0$ such that $\forall i \in N$, $v_i = v_j$.

Approach

- We develop algorithm RestrictedGoods which finds an EFX + PO allocation for RMG instances with goods only, by modifying the envy-cycle elimination algorithm.
- The following algorithm can be modified for RMG instances with IDO bads to allocate the bads in IDO order and return an EFX allocation.
- The allocation of $M^-$ can be further modified for RMG instances with general bads to return a PropMX allocation.
- Our results generalize previous work on binary and identical mixed manna [Aleksandrov and Walsh ’19, ’20].

Algorithm

Algorithm: EFX + PO for RMG instances with identical bads

$M \rightarrow M^+ \uplus M^2 \uplus M^-$, where

- $j \in M^+ \implies \exists i \in N : v_{ij} > 0$.
- $j \in M^2 \implies \exists i \in N : v_{ij} = 0$.
- $j \in M^- \implies \forall i \in N : v_{ij} < 0$.

Phase 1: Allocate all items $j \in M^+$ under modified valuations $v'$ using algorithm RestrictedGoods:

$$v'_j = \begin{cases} v_j, & \text{if } v_j \geq 0; \\ 0, & \text{if } v_j < 0 \end{cases}$$

Phase 2: Allocate all items $j \in M^2$ to agents $i$ such that $v_{ij} = 0$.

Phase 3: Allocate all items $j \in M^-$ in decreasing order of disutility to an agent that is a sink in the envy-graph of the current allocation.